HIGH SCHOOL MATHEMATICS

HIGH SCHOOL MATHEMATICS

THIRD COURSE

TEACHERS' EDITION

UNIT TWO 1957

EXPONENTS
AND
LOGARITHMS

UNIVERSITY OF ILLINOIS
COMMITTEE ON
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Unit 2

EXPONENTS AND LOGARITHMS

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Students should understand the listed kinds of real numbers to the extent that they are able to give examples of each kind. They should know that the integral real numbers,

$$---,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ ---$$

form a subset of the real numbers (and that the integer 0 is neither positive nor negative). They should know that the rational real numbers are those numbers which are quotients of integral real numbers, and that these include the integral real numbers. Thus:

$$3 = \frac{3}{1},\ -5 = \frac{-5}{1},\ 0 = \frac{0}{1},\ \text{etc.}$$

Irrational real numbers are real numbers which, like $\sqrt{2}$ and π, are not quotients of real integers. An irrational number is, also, not a quotient of rational numbers.

* * *

Review the meaning of the word 'power' given in the Review Exercises of Unit 1, THIRD COURSE. A power is a number; it has an exponent which is a number and it has a base which is also a number. An exponential, on the other hand, is a name for a power; an exponential contains an exponent symbol and a base symbol. It is not the case that every name for a power contains an exponent symbol and a base symbol. For example, '8^2' is an exponential which is a name for the second power of 8. The symbol '64' is another name for this power, but this name does not contain an exponent symbol nor does it contain a base symbol.

You know that when an exponent is a counting number, you can interpret an exponential like '4^3' as '$4 \times 4 \times 4$'. That is, such an exponential is a name of the product of "4 taken as a factor 3 times". However, there is no similar interpretation for exponentials such as:

$$5^{-2} \qquad 6^0 \qquad 3^{\frac{1}{2}} \qquad 9^{\sqrt{2}} .$$

For example, what could it mean to "take 3 as a factor $\frac{1}{2}$ times"? In this unit we shall extend the definition of exponentials to include cases in which the exponents are real numbers of the following kinds:

positive integers
zero
negative integers
rational numbers
and
irrational numbers.

In most cases we shall give recursive definitions; you will then use these definitions and mathematical induction to prove theorems about operating with such powers.

2.01 <u>Positive real integers</u>.--As you might guess, the recursive definition for powers with positive integral exponents is just like the one you constructed in Exercise 7 on page 1-53 of Unit 1.

For every real number a,

$$a^1 = a$$

and, for every integer $x > 0$,

$$a^{x+1} = a^x \cdot a.$$

Hence, in view of (a) and (b), it follows from the principle of mathematical induction for positive (real) integers that the property in question holds for all positive integers.

* * *

The theorem in Exercise 3 can be generalized to:

For every real number a, and
for every positive integer x,

$$(-a)^{2x-1} = -(a^{2x-1}).$$

3. The property is that expressed by:

$$(-2)^{2 \cdot \ldots - 1} = -(2^{2 \cdot \ldots - 1}).$$

(a) <u>1 has the property.</u>

$$(-2)^{2 \cdot 1 - 1} = (-2)^1 = -2$$

$$-(2^{2 \cdot 1 - 1}) = -(2^1) = -(2) = -2$$

(b) <u>The property is hereditary.</u>

Suppose that, for a given integer $k > 0$,

$$(-2)^{2k - 1} = -(2^{2k - 1}).$$

Then, for that k,

$$\begin{aligned}
(-2)^{2(k+1) - 1} &= (-2)^{2k + 2 - 1} \\
&= (-2)^{2k + 1} \\
&= (-2)^{2k} \cdot (-2) \\
&= [(-2)^{2k - 1} \cdot (-2)] \cdot (-2) \quad \left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right\} \text{recursive definition} \\
&= (-2)^{2k - 1} \cdot 4 \\
&= -(2^{2k - 1}) \cdot 4 \qquad \text{[inductive hypothesis]} \\
&= -[(2^{2k - 1} \cdot 2) \cdot 2] \\
&= -[2^{2k} \cdot 2] \\
&= -(2^{2k + 1}) \quad \left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right\} \text{recursive definition} \\
&= -(2^{2(k + 1) - 1}).
\end{aligned}$$

So, for every integer $k > 0$, if $(-2)^{2k - 1} = -(2^{2k - 1})$, then $(-2)^{2(k + 1) - 1} = -(2^{2(k + 1) - 1})$.

(continued on T. C. 2D)

B. Remind students that in any proof using mathematical induction they must be prepared to state which principle of mathematical induction they are using.

1. The property is that expressed by:

$$0^{\cdots} = 0.$$

(a) <u>1 has the property.</u>

By the recursive definition, $0^1 = 0$.

(b) <u>The property is hereditary.</u>

Suppose that, for a given integer $k > 0$, $0^k = 0$.
Then, for that k,

$$\begin{aligned} 0^{k+1} &= 0^k \cdot 0 && \text{[recursive definition]} \\ &= 0. \end{aligned}$$

So, for every integer $k > 0$, if $0^k = 0$ then $0^{k+1} = 0$.

Hence, in view of (a) and (b), it follows from the principle of mathematical induction for positive (real) integers that the property in question holds for all positive integers.

2. For every integer $x > 0$,

$$\begin{aligned} (-1)^{x+2} &= (-1)^{x+1} \cdot (-1) \\ &= [(-1)^x \cdot (-1)] \cdot (-1) \end{aligned} \Bigg\} \text{ recursive definition}$$

$$\begin{aligned} &= (-1)^x \cdot [(-1) \cdot (-1)] \\ &= (-1)^x \cdot [+1] \\ &= (-1)^x. \end{aligned}$$

[Students may give an inductive proof.]

(continued on T. C. 2C)

The step from '$1^z \cdot 1$' to '$1 \cdot 1$' is justified by the inductive hypothesis '$1^z = 1$'. In reaching the conclusion in this proof ask the students to state the principle of mathematical induction involved. It is the principle of mathematical induction for the positive real integers.

* * *

Exercises.

A. Be sure that students actually use the recursive definition given on page 2-1 to handle each of the exercises in Part A.

1. $4^3 = 4^2 \cdot 4 = (4^1 \cdot 4) \cdot 4 = (4 \cdot 4) \cdot 4$

2. $8^4 = 8^3 \cdot 8 = (8^2 \cdot 8) \cdot 8 = [(8^1 \cdot 8) \cdot 8] \cdot 8 = [(8 \cdot 8) \cdot 8] \cdot 8$

Point out to students that they can assert that $4^3 = 4 \cdot 4 \cdot 4$ because the associative principle for multiplication permits them to assert that $(4 \cdot 4) \cdot 4 = 4 \cdot (4 \cdot 4)$.

* * *

(continued on T. C. 2B)

As an example of how this recursive definition can be used we shall prove the following theorem:

For every integer $x > 0$, $1^x = 1$.

We need a principle of mathematical induction. Since the set in question is the set of positive integers, the principle we shall use is:

Every property of positive integers
which holds for +1 and is hereditary
holds for every positive integer.

Proof:

(a) 1 has the property.

$1^1 = 1$ by the recursive definition.

(b) The property is hereditary.

We want to prove that for every integer $z > 0$, if $1^z = 1$ then $1^{z+1} = 1$. By the recursive definition, for every integer $z > 0$,

$$\begin{aligned} 1^{z+1} &= 1^z \cdot 1 \\ &= 1 \cdot 1 \qquad \text{[Why?]} \\ &= 1 \end{aligned}$$

Thus, it follows from (a) and (b) and the principle of mathematical induction that for every integer $x > 0$, $1^x = 1$.

EXERCISES

A. Use the recursive definition to prove each of the following.

1. $4^3 = (4 \cdot 4) \cdot 4$
2. $8^4 = [(8 \cdot 8) \cdot 8] \cdot 8$

B. Use the recursive definition (and mathematical induction when necessary) to prove each of the following:

1. For every integer $x > 0$, $0^x = 0$.
2. For every integer $x > 0$, $(-1)^{x+2} = (-1)^x$.
3. For every integer $x > 0$, $(-2)^{2x-1} = -(2^{2x-1})$.

For every integer $x > 0$,
for every integer $y > 0$, and
for every real number a,

$$a^x \cdot a^y = a^{x+y}.$$

This is, of course, merely a re-wording of the boxed theorem on page 2-3.

B. 5. Using the theorem in Exercise 2 of Part B we have:

$$\begin{aligned}(-1)^4 \times (-1)^3 &= (-1)^2 \times (-1)^1 \\ &= (-1)^2 \times (-1) \\ &= (-1)^3 \\ &= (-1)^5 \\ &= (-1)^7.\end{aligned}$$

7. Using the theorem in Exercise 1 of Part B we have:

$$\begin{aligned}0^3 \times 0^5 &= 0 \times 0 \\ &= 0 \\ &= 0^8.\end{aligned}$$

8. For every real number k.

$$\begin{aligned}k^3 \times k^2 \times k^1 &= k^3 \times (k^1 \cdot k) \times k \\ &= (k^3 \times k) \times k \times k \\ &= k^4 \times k \times k \\ &= k^5 \times k \\ &= k^6.\end{aligned}$$

* * *

Notice that the inductive proof for the theorem in Part C is carried out "with respect to 'y' ". An inductive proof could also be carried out with respect to 'x'. The latter would, strictly speaking, prove the theorem:

(continued on T. C. 3B)

Sample. $5^2 \times 5^4 = 5^6$

Solution.
$$\begin{aligned}5^2 \times 5^4 &= (5^1 \times 5) \times 5^4\\ &= (5 \times 5) \times 5^4\\ &= 5 \times (5 \times 5^4)\\ &= 5 \times (5^4 \times 5)\\ &= 5 \times 5^5\\ &= 5^5 \times 5\\ &= 5^6\end{aligned}$$

[Note: There are other proofs for the Sample.]

4. $4^2 \times 4^3 = 4^5$

5. $(-1)^4 \times (-1)^3 = (-1)^7$

6. $2^3 \times 2^1 = 16$

7. $0^3 \times 0^5 = 0^8$

8. For every real number k, $k^3 \times k^2 \times k^1 = k^6$.

C. Exercises 4-8 of Part B suggest the following theorem:

> For every integer $y > 0$,
> for every integer $x > 0$, and
> for every real number a,
> $$a^x \cdot a^y = a^{x+y}.$$

We use mathematical induction to prove this theorem. Note that the property in question is that expressed by:

for every integer $x > 0$ and for every real number a,
$$a^x \cdot a^{\cdots} = a^{x+\cdots}.$$

Proof:

(a) The property holds for 1.

For every integer $x > 0$ and for every a,
$$\begin{aligned}a^x \cdot a^1 &= a^x \cdot a\\ &= a^{x+1}\end{aligned}$$

(b) The property is hereditary.

For every integer $x > 0$, for every a, and for every integer $z > 0$,

$$\begin{aligned}
\text{if} \quad a^x \cdot a^z &= a^{x+z}, \\
\text{then } a^x \cdot a^{z+1} &= a^x \cdot (a^z \cdot a) \\
&= (a^x \cdot a^z) \cdot a \\
&= a^{x+z} \cdot a \qquad \text{[Why?]} \\
&= a^{(x+z)+1} \\
&= a^{x+(z+1)}.
\end{aligned}$$

Hence, by (a), (b), and the principle of mathematical induction (for positive integers), the theorem follows.

This theorem we have proven is a case of what is often called the addition rule for exponents which may be stated as follows:

> The product of two powers of the same base is a power of that base with exponent equal to the sum of the exponents of the given powers.

[Notice that this statement does not restrict the kind of number used as exponent. We have only proved that the rule is valid in the case of positive integral exponents. Later, we shall prove that the rule applies to other kinds of exponents.] Note well in the statement of the addition rule for exponents the phrases the same base and the product.

Here is an instance of the addition rule for exponents:

$$3^5 \times 3^7 = 3^{12}.$$

Notice that

$$3^5 \times 2^7 \neq 6^{12}$$

and

$$3^5 + 3^7 \neq 3^{12}.$$

25. $(x + 3)^6$

26. $(y + 1)^5$

27. $(z - 2)^6$

28. $(a + b)^8$

29. $(x - 1)^5(x + 5)^2$

30. $(a + 4)^9(a + 1)^6$, or: $(a + 1)^6(a + 4)^9$

31. $(3y^2 + y + 7)^8$

32. $[(a + b)^8]^9$

33. $(2x^2 + 9x + 4)^{13}$

34. In this exercise students need to use the generalization of the theorem in Exercise 3 of Part B. This generalization implies the following:

For every a and b,

$$(a - b)^3 = -(b - a)^3.$$

If students have not yet seen the generalization of the theorem in Exercise 3 they should do so at this time.

35. This exercise needs to be handled in the same manner as Exercise 34.

36. Be sure that students understand the need for the restriction on the variable 'x'.

C. (Cont.)

1. 5^8
2. 5^{a+2}
3. 5^{a+b}
4. 6^{a+b}
5. x^{a+b}
6. $x^a y^b$
7. 19^{a+b+c}
8. a^9
9. $a^5 + a^4$
10. 1
11. 2
12. 17
13. s^{a+b+c}
14. 3^{a+b+7}, or: 3^{a+7+b}, etc.
15. 4^{5+b}, x^{a+c}
16. $x^c y^b z^a a^d$
17. 6^{a+b+c}
18. x^{3a}
19. $a^{x+6} \cdot b^{y+2}$
20. $t^{r+s} \cdot r^t \cdot s^t$
21. Some students may give:

$$x^{c+d} y^{c+d}$$

as an answer rather than:

$$(xy)^{c+d}.$$

Of course, those students would be assuming a theorem that is yet to be proved (see page 2-7).

22. $(abc)^{x+y}$
23. $x^{a+b+c+d}$
24. Be sure that students recognize the difference in handling expressions such as:

$$aa^2$$

and:

$$(aa)^2.$$

(continued on T. C. 5B)

Simplify each of the following. [The domain of each exponent pronumeral is the set of positive integers.]

1. $5^6 \cdot 5^2$
2. $5^a \cdot 5^2$
3. $5^a \cdot 5^b$
4. $6^a 6^b$
5. $x^a x^b$
6. $x^a y^b$
7. $19^a 19^b 19^c$
8. $a^2 a^3 a^4$
9. $a^2 a^3 + a^4$
10. $1^{364} 1^{25} 1^{321}$
11. $1^{364} + 1^7$
12. $3^2 + 2^3$
13. $s^a s^b s^c$
14. $3^a 3^7 3^b$
15. $4^5 x^a 4^b x^c$
16. $x^c y^b z^a a^d$
17. $6^b 6^a 6^c$
18. $x^a x^a x^a$
19. $a^x b^y b^2 a^6$
20. $t^r r^t t^s s^t$
21. $(xy)^c (xy)^d$
22. $(abc)^x (abc)^y$
23. $x^{a+b} x^{c+d}$
24. $aa^2 a^x a^{y+z}$
25. $(x + 3)^4 (x + 3)^2$
26. $(y + 1)^2 (y + 1)^3$
27. $(z - 2)(z - 2)^5$
28. $(a + b)^2 (a + b)(a + b)^5$
29. $(x + 1)^5 (x + 5)^2$
30. $(a + 4)^2 (a + 1)(a + 4)^7 (a + 1)^5$
31. $(3y^2 + y + 7)^5 (3y^2 + y + 7)^3$
32. $[(a + b)^8]^2 \cdot [(b + a)^8]^7$
33. $(2x^2 + 9x + 4)^3 (2x^2 + 9x + 4)^9 (2x + 1)(x + 4)$
34. $(a - b)^3 (b - a)^2$
35. $(3x - 2y)^7 (2y - 3x)^5$
36. $\left[\frac{1}{x + 1}\right]^2 \left[\frac{1}{x + 1}\right]^7$, $[x \neq -1]$
37. $\left[\frac{k + 5}{m - 1}\right]\left[\frac{k + 5}{m - 1}\right]^4$, $[m \neq 1]$

(continued on next page)

[2-6]

C. (Cont.)

38. If a student gives as his simple expression: $\frac{1}{x^6}$, point out to him that he is making use of a theorem which has not yet been proved.

41. Be sure students understand why the restriction:

$$xy \neq 0$$

is equivalent to the restriction:

$$x \neq 0 \text{ and } y \neq 0.$$

* * *

D. Although such proofs are not in accord with the instructions for this part, students should recognize that the statement in Exercise 2 can be proved by applying the theorem given at the top of page 2-2 and that the statement in Exercise 3 can be proved by applying the theorem in Exercise 1 of Part B.

6. For every t, and for every integer x > 0,

$$\begin{aligned}
(t^x)^4 &= (t^x)^3 \cdot t^x \\
&= [(t^x)^2 \cdot t^x] \cdot t^x \\
&= (t^x)^2 \cdot t^{2x} \\
&= [(t^x)^1 \cdot t^x]\, t^{2x} \\
&= t^x \cdot t^{3x} \\
&= t^{4x}.
\end{aligned}$$

38. $\left[\frac{1}{x}\right]^2\left[\frac{1}{x}\right]^3\left[\frac{1}{x}\right]$, $[x \neq 0]$

39. $\frac{1}{x} + \frac{1}{x}$, $[x \neq 0]$

40. $\frac{1}{a^b} \cdot \frac{1}{a^c}$, $[a \neq 0]$

41. $\frac{1}{3x^5y^6} \cdot \frac{1}{3^2x^3y^7}$, $[xy \neq 0]$

D. Use the recursive definition and the theorem in Part C to prove each of the following.

Sample. $(5^2)^3 = 5^6$

Solution.
$$\begin{aligned}(5^2)^3 &= (5^2)^{2+1} \\ &= (5^2)^2 \cdot 5^2 \\ &= [(5^2)^1 \cdot 5^2] \cdot 5^2 \\ &= [5^2 \cdot 5^2] \cdot 5^2 \\ &= 5^4 \cdot 5^2 \\ &= 5^6\end{aligned}$$

1. $(8^2)^4 = 8^8$

2. $(1^5)^2 = 1^{10}$

3. $(0^4)^8 = 0^{32}$

4. $[(-9)^3]^5 = (-9)^{15}$

5. For every t, $(t^3)^4 = t^{12}$.

6. For every t, and for every integer $x > 0$, $(t^x)^4 = t^{4x}$.

E. The exercises in Part D suggest the following theorem:

> For every integer $y > 0$, for every integer $x > 0$, and for every real number a,
>
> $$(a^x)^y = a^{xy}.$$

orem.

Before we can use the recursive definition in the proof of the theorem in Part E, we must be sure that

> For every integer $x > 0$, and for every real number a,
>
> a^x is a real number.

We can prove the boxed statement by the principle of mathematical induction for the set of positive integers. We also need to assume that the set of real numbers is closed under multiplication, that is, the product of a real number by a real number is a real number.

Proof:

(a) 1 has the property.

For every real number a,

$a^1 = a$ and a is a real number.

(b) The property is hereditary.

For every real number a and for every integer $z > 0$,

$a^{z+1} = a^z \cdot a$ [recursive definition]

a^z is a real number [inductive hypothesis]

a is a real number

$a^z \cdot a$ is a real number [closure under multiplication]

Hence, it follows from (a) and (b) and the principle of mathematical induction for the set of positive integers that

For every integer $x > 0$, and for every real number a,

a^x is a real number.

`orem.

(a) y^6 (b) 3^7 (c) x^4 (d) 2^4, or: 16

(e) $(3a)^6$ (f) $(xy)^{10}$ (g) $(6 + a)^8$ (h) $(\sqrt{2})^{12}$

* * *

F. 5. Again point out that the exponent symbol affects only that to which it is directly attached. In other words, contrast '$16a^4$' with '$(16a)^4$'. Compare this with the case of the '-' in '$-2 \cdot 3$'. The latter symbol is taken as a name for the product of -2 and 3.

* * *

G. [We give a sketch of the proof.]

(a) $(ab)^1 = ab = a^1b^1$

(b) $(ab)^{x+1} = (ab)^x(ab)$

$= (a^xb^x)(ab)$ [inductive hypothesis]

$= (a^xa)(b^xb)$

$= a^{x+1}b^{x+1}$

orem.

E. 1. Property is that expressed by:

for every integer $x > 0$, and
for every real number a,

$$(a^x)^{\cdots} = a^{x \cdot \cdots}.$$

(a) 1 has the property.

For every integer $x > 0$, and for every real number a,

$$(a^x)^1 = a^x \qquad \text{[recursive definition]}$$
$$= a^{x \cdot 1} \qquad \text{[principle of 1]}$$

(b) The property is hereditary.

Suppose that, for a given integer $k > 0$,

$$(a^x)^k = a^{xk}.$$

Then, for that k,

$$(a^x)^{k+1} = (a^x)^k \cdot a^x \qquad \text{[recursive definition]}$$
$$= a^{xk} \cdot a^x$$
$$= a^{xk+x} \qquad \text{[addition rule]}$$
$$= a^{x(k+1)}.$$

So, for every integer $k > 0$, if $(a^x)^k = a^{xk}$ then $(a^x)^{k+1} = a^{x(k+1)}$.

Hence, in view of (a) and (b), it follows from the principle of mathematical induction for positive integers that the property in question holds for every positive integer.

2. [Forestall attempts to give an answer such as '$x^{10}y^{10}$' for Exercise (f). This answer depends on the theorem to be proved in Part G.]

(continued on T. C. 7B)

1. Use the proof in Part C as a model in proving the above theorem.

   This theorem is a case of what is often called the multiplication rule for exponents.

2. Use the multiplication rule for exponents to simplify each of the following.

   (a) $(y^3)^2$ (b) $(3^7)^1$ (c) $(x^1)^4$ (d) $(2^2)^2$

   (e) $[(3a)^2]^3$ (f) $[(xy)^2]^5$ (g) $[(6 + a)^4]^2$ (h) $[(\sqrt{2})^3]^4$

F. Use the recursive definition and the addition rule for exponents to prove each of the following.

Sample. $(2 \cdot 5)^3 = 2^3 \cdot 5^3$

Solution.
$$\begin{aligned}(2 \cdot 5)^3 &= (2 \cdot 5)^2 \cdot (2 \cdot 5)\\ &= [(2 \cdot 5)^1 \cdot (2 \cdot 5)] \cdot (2 \cdot 5)\\ &= [(2 \cdot 5) \cdot (2 \cdot 5)] \cdot (2 \cdot 5)\\ &= (2 \cdot 2 \cdot 2) \cdot (5 \cdot 5 \cdot 5)\\ &= 2^3 \cdot 5^3\end{aligned}$$

1. $(3 \cdot 8)^2 = 3^2 \cdot 8^2$
2. $[4(-3)]^2 = 4^2(-3)^2$
3. $(14)^3 = 7^3 \cdot 2^3$
4. $(-\sqrt{2})^5 = \pi^5(\sqrt{2})^5$
5. For every real number a, $(2a)^4 = 16a^4$.

G. Use mathematical induction to prove the theorem suggested by the exercises in Part F.

For every integer $x > 0$, and for all real numbers a and b, $(ab)^x = a^x b^x$.

en-

$$
\begin{aligned}
(2 \times 4 \times 7)^3 &= ((2 \times 4) \times 7)^3 \\
&= (2 \times 4)^3 \times 7^3 \\
&= 2^3 \times 4^3 \times 7^3.
\end{aligned}
$$

Of course, after a small amount of practice, students will be able to omit the intermediate steps.

* * *

H. 1. $a^5b^5c^5$ 2. $16y^4z^4$ 3. $a^6(m - 1)^6$

4. $(x + 1)^4(x + 2)^4$ 5. $a^m b^m c^m d^m$ 6. $x^4y^8z^{10}$

7. $3^z a^{2z} b^{xz} c^{2yz}$ 8. $10ab^2c^3$ 9. $125\pi^3r^3$

en-

G. (Cont.)

The distributive principle with which the students are already acquainted is called the distributive principle for multiplication over addition. An instance of a "distributive principle for exponentiation over addition" is:

$$(3 + 2)^2 = 3^2 + 2^2 .$$

Obviously, this statement is false. However, it illustrates an error commonly committed by students who, apparently, deduce such a principle from the distributive principle for multiplication over addition and the distributive principle for exponentiation over multiplication.

* * *

In simplifying such an expression as '$3^5 \cdot 3^2 \cdot 3^9$' it should be considered as an abbreviation for '$(3^5 \cdot 3^2) \cdot 3^9$'. Thus,

$$\begin{aligned} 3^5 \cdot 3^2 \cdot 3^9 &= (3^5 \cdot 3^2) \cdot 3^9 \\ &= 3^7 \cdot 3^9 \\ &= 3^{16}. \end{aligned}$$

Similarly,

$$\begin{aligned} 3^{5+2+9} &= 3^{(5+2)+9} \\ &= 3^{5+2} \cdot 3^9 \\ &= 3^5 \cdot 3^2 \cdot 3^9, \\ [(5^2)^3]^4 &= [5^6]^4 \\ &= 5^{24}, \end{aligned}$$

(continued on T. C. 8B)

This theorem could be called the <u>distributive principle for exponentiation over multiplication</u>. Do you see why this name could be appropriate? Does the "distributive principle for exponentiation over addition" hold?

H. Give equivalent expressions which contain fewer grouping symbols.

Sample. $(x^2y^3z^4)^5$

Solution. For every x, y, and z,

$$\begin{aligned}(x^2y^3z^4)^5 &= (x^2y^3)^5(z^4)^5\\ &= (x^2)^5(y^3)^5(z^4)^5\\ &= x^{10}y^{15}z^{20}\end{aligned}$$

[In practice you can omit most of the steps.]

1. $(abc)^5$
2. $(2yz)^4$
3. $[a(m - 1)]^6$
4. $[(x + 1)(x + 2)]^4$
5. $(abcd)^m$
6. $(x^2y^4z^5)^2$
7. $(3a^2b^xc^{2y})^z$
8. $(10ab^2c^3)^1$
9. $(5\pi r)^3$

I. Simplify. [Leave answers in exponential form.]

Sample 1. $2^3 4^5 2^4$

Solution.

$$\begin{aligned}2^3 4^5 2^4 &= 2^7 4^5\\ &= 2^7(2^2)^5\\ &= 2^7 2^{10}\\ &= 2^{17}\end{aligned}$$

Sample 2. $3^2 2^3 30^4$

Solution.

$$\begin{aligned}3^2 2^3 30^4 &= 3^2 2^3(2 \cdot 3 \cdot 5)^4\\ &= 3^2 2^3 2^4 3^4 5^4\\ &= 2^7 3^6 5^4\end{aligned}$$

1. 3^8

2. $2^9 \cdot 5^6$, or: $10^6 \cdot 2^3$

3. $3^3 \cdot 5^8$

4. $2^{27} \cdot 3^6$

5. $2^8 \cdot 3^5 \cdot 17^7$

6. $2^{m+2} \cdot 3^2$

7. 12^m, or: $2^{2m} \cdot 3^m$

8. $2^{3m+4} \cdot 3^{m+4}$

9. $2^5 \cdot 3^6 \cdot x^9$

10. $2^{8m} \cdot 3^m \cdot x^m \cdot y^m$

11. $(2x)^{m(p+m)}$

12. 3^{10}

13. 5^{20}

14. 7^4

15. $3^{12} \cdot 2^{18}$, or: $6^{12} \cdot 2^6$

16. x^6y^{15}

17. 2^{30}

18. $x^4y^8z^{12}$

19. $y^{3m}m^{3m}$

20. $t^{r(rt)}s^{t(rt)}$

21. 3^{12}

22. $5^{12} \cdot 2^{16}$

23. $5^{20} \cdot 2^{18}$

24. $2^{32} \cdot 3^{18}$

25. $a^{10}b^5$

26. $x^{14}y^7z^5$

27. $x^{am+b}y^{am+c}$

28. $x^{a+2b}y^{a+2b}z^{a+2b}$

29. $(x+1)^{10}y^9$

30. $x^{12}y^{12}$

31. $(t+1)^{2t+3}$

32. $y^{2st}z^{rs+s}$

33. $(1-x)^{3x+3}$

1. $9^2 3^4$
2. $(10)^3(20)^3$
3. $15^2 \cdot 3 \cdot 5^6$
4. $48^6 2^3$
5. $17^5 51^2 6^3 2^5$
6. $2^m 6^2$
7. $2^m 6^m$
8. $12^m 2^m 6^4$
9. $(2x)^3(6x)^2(3x)^4$
10. $(16x)^m(48y)^m$
11. $(x^p 2^m)^m(x^m 2^p)^m$
12. $(3^2)^5$
13. $(5^4)^5$
14. $(7^2)^2$
15. $(3^2 2^3)^6$
16. $(x^2 y^5)^3$
17. $(2^2 4^3 2^2)^3$
18. $(xy^2 z^3)^4$
19. $(ym)^{3m}$
20. $(t^r s^t)^{rt}$
21. $(3^2 3^4)^2$
22. $(5^1 20^2)^4$
23. $(200^2 50^3)^2$
24. $(3^4 12^2)^2(4^3 6^2)^3$
25. $(a^2 b)^2(ba^2)^3$
26. $(x^2 y)^4(yz)^3(x^3 z)^2$
27. $(x^a y^a)^m x^b y^c$
28. $(xyz)^a(x^2 y^2 z^2)^b$
29. $[(x+1)^3 y^2]^2(x+1)^4 y^5$
30. $[(x^3)^2(y^2)^3]^2$
31. $(t+1)^{t+1}(t+1)^{t+2}$
32. $(y^t z^r)^s \cdot y^{st} \cdot z^s$
33. $(1-x)^{x+1}(x-1)^{2x+2}$

2.02 <u>The real integer 0 as an exponent.</u>--In the exercises of the preceding section the following three theorems were proved:

(I) For every integer $y > 0$,
for every integer $x > 0$, and
for every real number a,

$$a^x \cdot a^y = a^{x+y}.$$

(II) For every integer $y > 0$,
for every integer $x > 0$, and
for every real number a,

$$(a^x)^y = a^{xy}.$$

(III) For every integer $x > 0$, and
for all real numbers a and b,

$$(ab)^x = a^x b^x.$$

Line 11.

The word 'domain' may be new to THIRD COURSE students. Define it for them as follows:

> The domain of a pronumeral (or numerical variable) is a set of numbers. The pronumeral holds a place for a name of any one of these numbers.

* * *

The reason why we use the old recursive definition to provide a clue for a new recursive definition is that we want the old definition to be "included" in the new one.

* * *

Equation 5 does not give us a clue to the definition of '0^0' because equation 5 is satisfied when $0^0 = 0$, $0^0 = 9$, $0^0 = 2\pi$, etc.

In proving these theorems we used the principle of mathematical induction for positive integers, and the following recursive definition:

For every real number a,

$$a^1 = a$$

and, for every integer $x > 0$,

$$a^{x+1} = a^x \cdot a.$$

The next step in our program of discussing all real number exponents is to define exponentials such as:

2^0 $\quad (\frac{1}{3})^0$ $\quad (-7)^0$ $\quad \pi^0$ $\quad (\sqrt{2})^0$ $\quad 0^0$

It would be convenient if the new definition could enable us to prove new theorems very much like (I), (II), and (III) above in which the domain of 'x' and of 'y' includes the integer 0 as well as the positive integers. A clue to the construction of such a definition can be found by considering the recursive definition given above.

That definition tells us that for every a and for every integer $x > 0$,

$$(1) \quad a^{x+1} = a^x \cdot a.$$

If we replace 'x' by '0' we get the equation:

$$(2) \quad a^{0+1} = a^0 \cdot a.$$

Since $a^{0+1} = a^1 = a$, equation (2) is equivalent to:

$$(3) \quad a = a^0 \cdot a.$$

For every $a \neq 0$, equation (3) holds if and only if

$$(4) \quad a^0 = 1.$$

For $a = 0$, equation (3) becomes:

$$(5) \quad 0 = 0^0 \cdot 0.$$

Clearly, equation (5) does not give us a clue to a definition of '0^0' [Why?]. However, if we consider the addition rule for exponents [See page 2-4] for the case in which the base is 0 and each exponent is 0, we get:

$$0^0 \cdot 0^0 = 0^{0+0}$$

In defining '0^0' we depart from the treatment usually found in conventional textbooks. There, '0^0' is not defined.

* * *

Students should compare this new recursive definition with the old one as given on page 2-10. It is easy to derive from this new definition the theorem that for every a, $a^1 = a$ by replacing 'x' in '$a^{x+1} = a^x \cdot a$' by '0'. There may be a tendency for students to feel that such a proof involves a certain amount of circularity in view of equation 2 on page 2-10. However, the work on page 2-10 was largely heuristic, that is, suggestive of the form of the new definition. Actually, we could have stated the new definition without any build-up at all.

* * *

A. The proofs of the theorems similar to theorems 1, 2 and 3 are entirely like those given for the older theorems except that, instead of demonstrating that 1 has the property in question, one must demonstrate that 0 has the property in question. Students will not suffer from giving new proofs in spite of the fact that such proofs involve repetition of earlier ones.

B. 1. 1   2. 0   3. 4

4. 0   5. 1   6. -1

or:

$$(6) \quad 0^0 \cdot 0^0 = 0^0$$

If (6) is to be true, then '0^0' must be a name for a root of the equation:

$$(7) \quad p \cdot p = p$$

Equation (7) has the roots 0 and 1. Hence, if (6) is to be true, '0^0' must be a name for 0 or a name for 1. It would be more convenient to say that $0^0 = 1$, for this alternative would result in a single definition for powers with 0 exponent.

We now state a new recursive definition.

> For every real number a,
>
> $$a^0 = 1$$
>
> and, for every integer $x \geq 0$,
>
> $$a^{x+1} = a^x \cdot a.$$

[Note: You can derive from the above definition the theorem that, for every a, $a^1 = a$. Do it.]

EXERCISES

A. Refer to theorems (I), (II), and (III) on page 2-9. State new theorems similar to these but in which the domain of 'x' and of 'y' is the set of non-negative integers. Then use the new recursive definition and the principle of mathematical induction for non-negative integers to prove the new theorems.

B. Use the theorems you have just proved and the recursive definition to write the simplest non-exponential names for the powers listed below.

1. 2^0	2. 0^2	3. $5^0 \cdot 2^2$
4. $0^5 \cdot 2^2$	5. $(-6)^0$	6. $-(-6)^0$

(continued on next page)

?-12]

B. (Cont.)

7. -1	8. 1	9. 1
10. 1	11. 1	12. 1
13. 1	14. 1	15. 1
16. 1	17. 1	18. 1
19. 1	20. 1	21. 1

C.

1. a^x	2. a^y	3. 1
4. 1	5. 1	6. 1
7. 1	8. 1	9. -1

7. $-(+6)^0$ 8. $(3 \cdot 9)^0$ 9. $(5\pi + 2)^0$

10. $\left[\frac{163}{497}\right]^0$ 11. $\frac{3^0}{4^0}$ 12. $\left[\frac{3}{4}\right]^0$

13. $5^0 8^0$ 14. $(5 \cdot 8)^0$ 15. $5^0 \cdot 5^0$

16. $(3^0)^4$ 17. $(3^3)^0$ 18. $(3^0)^0$

19. $(15 - 5 \cdot 3)^0$ 20. 0^0 21. $(6 - 9)^0$

C. Simplify.

1. $a^x \cdot a^0$ 2. $a^0 \cdot a^y$ 3. $a^0 \cdot a^0$

4. $(a^x)^0$ 5. $(a^0)^x$ 6. $(a^0)^0$

7. $(ab)^0$ 8. $(-ab)^0$ 9. $-(ab)^0$

2.03 Negative real integers as exponents.--Our experience with constructing a recursive definition to include the case of the 0 exponent suggests that we use a similar approach in deciding upon a definition of exponentials which will include the case of a negative integer exponent.

As before, we want the equation:

$$(1) \quad a^{x+1} = a^x \cdot a$$

to hold for every a and for every integer x. Let us replace in (1) the symbol 'x' by '-1'. We get:

$$a^{-1+1} = a^{-1} \cdot a$$

or:

$$a^0 = a^{-1} \cdot a$$

or:

$$(2) \quad 1 = a^{-1} \cdot a.$$

Now, (2) holds for every $a \neq 0$ if and only if

$$(3) \quad a^{-1} = \frac{1}{a}.$$

In case $a = 0$, equation (2) becomes:

$$(4) \quad 1 = 0^{-1} \cdot 0.$$

Stress line three: Students should realize that we do not define the expression '0^{-1}' because it would lead to a contradication of the principle of 0. [See equation (5).] And not because it is too difficult to do so.

* * *

Instead of the boxed recursive definition we could have given as an alternative, explicit, defining principle:

For every a not equal to 0,
and every integer $n < 0$,

$$a^n = \frac{1}{a^{-n}} .$$

Our reasons for not doing this are that

(1) it necessitates considering numerous cases in proving theorems [for example: $(a^x)^y = a^{x \cdot y}$ $\begin{cases} x \geq 0, & y < 0 \\ x \leq 0, & y \geq 0 \\ x < 0, & y \leq 0 \end{cases}$],

(2) we prefer to emphasize mathematical induction proofs and recursive definitions are useful in such proofs,

(3) it is more aesthetically satisfying to stick to one kind of definition.

Stress the fact that exponentials with base 0 and negative exponents are <u>not</u> defined.

* * *

Since, for the set of integers < 0, the follower of x is $x - 1$, it is natural to state the second part of the recursive definition as we have done.

If (4) is to be true, then '0^{-1}' must be a name for a root of the equation:

$$(5) \quad 1 = p \cdot 0 .$$

Clearly, (5) has no roots. Hence, we do not define '0^{-1}'.

Now, let us replace in (1) the symbol 'x' by '-2':

$$a^{-2+1} = a^{-2} \cdot a$$

or:

$$(6) \quad a^{-1} = a^{-2} \cdot a$$

If equation (3) holds for every $a \neq 0$ then equation (6) holds for every $a \neq 0$ if and only if

$$(7) \quad a^{-2} = \frac{1}{a^2} .$$

Similarly, we can show that, for every $a \neq 0$,

$$(8) \quad a^{-3} = \frac{1}{a^3}$$

if equation (7) holds for every $a \neq 0$.

We could consider, for every $a \neq 0$, powers such as a^{-4}, a^{-5}, a^{-6}, ... However, the few cases already considered suggest the following recursive definition:

For every real number $a \neq 0$

$$a^{-1} = \frac{1}{a}$$

and, for every integer $x < 0$,

$$a^{x-1} = a^x \cdot \frac{1}{a} .$$

We can use this definition to show, for example, that, for every $a \neq 0$,

$$a^{-4} = \frac{1}{a^4} .$$

Note that if we include in our new recursive definition 'for every $a \neq 0$, $a^0 = 1$' and (*) then we can derive:

$$\text{For every } a \neq 0,\ a^{-1} = \frac{1}{a}.$$

This derivation is given at the bottom of page 14. The derivation demonstrates that the boxed recursive definition on page 2-15 does, indeed, include the recursive definitions on pages 2-13 and 2-11.

Proof:

For every $a \neq 0$,

$$a^{-4} = a^{-3} \cdot \frac{1}{a} \qquad [\text{since } -4 = -3 - 1]$$

$$= (a^{-2} \cdot \frac{1}{a}) \cdot \frac{1}{a}$$

$$= a^{-2} \cdot \frac{1}{a^2}$$

$$= (a^{-1} \cdot \frac{1}{a}) \cdot \frac{1}{a^2}$$

$$= a^{-1} \cdot \frac{1}{a^3}$$

$$= \frac{1}{a} \cdot \frac{1}{a^3}$$

$$= \frac{1}{a^4}.$$

We seek, now, a recursive definition which will cover <u>all</u> integer exponents. The foregoing definition covers negative integer exponents, and the one on page 2-11 covers non-negative integer exponents. If we restrict ourselves to non-zero bases ($a \neq 0$) then the second part of each definition can be derived from:

(*) for every integer x,

$$a^{x+1} = a^x \cdot a$$

Also, if, for every $a \neq 0$, $a^0 = 1$, then (*) gives us:

$$a^{-1+1} = a^{-1} \cdot a$$

or:

$$a^0 = a^{-1} \cdot a$$

or:

$$1 = a^{-1} \cdot a$$

or:

$$a^{-1} = \frac{1}{a}.$$

Thus, the following recursive definition "combines" the two earlier ones and covers the case of 0 base:

> For every real number $a \neq 0$,
> $$a^0 = 1,$$
> and, for every integer x,
> $$a^{x+1} = a^x \cdot a;$$
> for every integer $x > 0$,
> $$0^x = 0;$$
> and
> $$0^0 = 1.$$

EXERCISES

A. Use the foregoing recursive definition for integral powers to prove each of the following statements.

Sample. $6^{-2} = \frac{1}{6^2}$

Solution. In '$a^{x+1} = a^x \cdot a$' replace 'x' by '-2' and 'a' by '6'. Then we have:

$$6^{-2+1} = 6^{-2} \cdot 6$$

or:

$$(1) \quad 6^{-1} = 6^{-2} \cdot 6$$

Now, we use the recursive definition to find the value of '6^{-1}':

$$6^{-1+1} = 6^{-1} \cdot 6,$$

and

$$6^0 = 1.$$

Hence,

$$1 = 6^{-1} \cdot 6,$$

or

$$(2) \quad 6^{-1} = \frac{1}{6}.$$

So, from (1) and (2) we get:

$$\frac{1}{6} = 6^{-2} \cdot 6$$

or:

$$6^{-2} = \frac{1}{6^2}.$$

1. $(-3)^{-3} = \dfrac{1}{(-3)^3}$

2. $7^4 = \dfrac{1}{7^{-4}}$

3. Why can't you apply the recursive definition to prove that $0^{-5} = \dfrac{1}{0^5}$?

B. You have proved the addition rule for exponents in the case of non-negative integral exponents [Part A on page 2-11]. Using the new recursive definition we shall now prove this rule for all integral powers.

> For every real number $a \neq 0$,
> and for all integers x and y,
>
> $$a^x \cdot a^y = a^{x+y}.$$
>
> For all non-negative integers x and y,
>
> $$0^x \cdot 0^y = 0^{x+y}.$$

We use mathematical induction to prove the first statement in the box. Note that the property in question is expressed by:

for every integer x and
for every real number $a \neq 0$,

$$a^x \cdot a^{\cdots} = a^{x+\cdots}.$$

solutions to the exercises in Part E and Part F on pages 2-19 and 2-20, respectively.

Note that in the given proof of (b_{II}) the replacement of 'a^{y-1}' by '$\left(a^y \cdot \frac{1}{a}\right)$' is justified by the recursive definition:

For every integer x, $a^{x+1} = a^x \cdot a$.

Hence, for the integer y in question,

$$\begin{aligned} a^y &= a^{(y-1)+1} \\ &= a^{y-1} \cdot a . \end{aligned}$$

Since $a \neq 0$, it follows that

$$a^{y-1} = a^y \cdot \frac{1}{a} .$$

Also, as in the proof of (b_I), the replacement of '$a^x \cdot a^y$' by 'a^{x+y}' is justified by the inductive hypothesis.

Proofs given for (b_I) and (b_{II}) are more abbreviated than those given in Unit 1 of Third Course. Here is a fuller treatment for (b_I):

Suppose that, for a given integer $y \geq 0$,

$$a^x \cdot a^y = a^{x+y}. \qquad \text{[Inductive hypothesis]}$$

By the recursive definition,

$$a^x \cdot a^{y+1} = a^x(a^y \cdot a)$$
$$= (a^x \cdot a^y) \cdot a.$$

So, by the inductive hypothesis, for the number y in question,

$$a^x \cdot a^{y+1} = a^{x+y} \cdot a.$$

Hence, by the recursive definition,

$$a^x \cdot a^{y+1} = a^{(x+y)+1} = a^{x+(y+1)}$$

Therefore, for every $y \geq 0$, if $a^x \cdot a^y = a^{x+y}$ then $a^x \cdot a^{y+1} = a^{x+(y+1)}$.

A similar development can be given for (b_{II}).

The abbreviated proofs given in the text may lead students to give the very theorem which is to be proved as a reason for going from the expression:

$$(a^x \cdot a^y) \cdot a$$

to:

$$(a^{x+y}) \cdot a \,.$$

They should, instead, cite the inductive hypothesis as a reason. Check your students' understanding of this point when you go over their

(continued on T. C. 17B)

We shall use <u>two</u> principles of mathematical induction.

(I) Every property of integers ≥ 0 which holds for 0 and is hereditary [y's follower is $y + 1$] holds for every integer ≥ 0.

(II) Every property of integers ≤ 0 which holds for 0 and is hereditary [y's follower is $y - 1$] holds for every integer ≤ 0.

<u>Proof</u>:

(a) <u>0 has the property.</u>

$$\begin{aligned} a^x \cdot a^0 &= a^x \cdot 1 \\ &= a^x \\ &= a^{x+0}. \end{aligned}$$

(b_I) <u>The property is hereditary for integers ≥ 0.</u>

We want to prove that for every integer $y \geq 0$, if $a^x \cdot a^y = a^{x+y}$ then $a^x \cdot a^{y+1} = a^{x+(y+1)}$.

$$\begin{aligned} a^x \cdot a^{y+1} &= a^x \cdot (a^y \cdot a) \\ &= (a^x \cdot a^y) \cdot a \\ &= (a^{x+y}) \cdot a \qquad \text{[Why?]} \\ &= a^{(x+y)+1} \\ &= a^{x+(y+1)}. \end{aligned}$$

(b_{II}) <u>The property is hereditary for integers ≤ 0.</u>

We want to prove that, for every $y \leq 0$, if $a^x \cdot a^y = a^{x+y}$ then $a^x \cdot a^{y-1} = a^{x+(y-1)}$.

$$\begin{aligned} a^x \cdot a^{y-1} &= a^x \cdot (a^y \cdot \frac{1}{a}) \\ &= (a^x \cdot a^y) \cdot \frac{1}{a} \\ &= (a^{x+y}) \cdot \frac{1}{a} \\ &= a^{(x+y)-1} \\ &= a^{x+(y-1)}. \end{aligned}$$

The second statement in the box can be proved as follows:

For every integer $x \geq 0$, if $y > 0$

then $0^x \cdot 0^y = 0^x \cdot 0 = 0 = 0^{x+y}$,

while $0^x \cdot 0^0 = 0^x \cdot 1 = 0^x = 0^{x-0}$.

* * *

In simplifying the expression in Sample 2 students may want to assert that $\frac{1}{5^{12}} = 5^{-12}$. Although this assertion is correct, it needs to be proved. The proof involves nothing more than a step very much like that taken in the next to the last line of the Solution. In Part D on page 2-19, the student is asked to prove two theorems the first of which covers the case in question here. Let him practice the device indicated in this Solution instead of using the theorems of Part D without proof.

Hence, by principle (I) it follows from (a) and (b_+) that for every $a \neq 0$, for every integer x, and for every integer $y \geq 0$,

$$a^x \cdot a^y = a^{x+y}.$$

By principle (II) it follows from (a) and (b_-) that for every $a \neq 0$, for every integer x, and for every integer $y \leq 0$,

$$a^x \cdot a^y = a^{x+y}.$$

Therefore, since for every integer y, either $y \geq 0$ or $y \leq 0$, we have proved the first statement in the box on page 2-17. <u>The student should prove the second statement in the box.</u>

C. Apply the theorem proved in Part B to simplify each of the following. Leave answers in simplest exponential form.

<u>Sample 1</u>. $a^{-2} \times a^{3} \times a^{-5}$

<u>Solution</u>.

$$\begin{aligned} & a^{-2} \times a^{3} \times a^{-5} \\ &= a^{-2+3} \times a^{-5} \\ &= a^{1} \times a^{-5} \\ &= a^{1-5} \\ &= a^{-4} \end{aligned}$$

<u>Sample 2</u>. $\dfrac{5^{-3} \times 5^{-7}}{5^{12}}$

<u>Solution</u>.

$$\begin{aligned} \frac{5^{-3} \times 5^{-7}}{5^{12}} &= \frac{5^{-10}}{5^{12}} \\ &= \frac{5^{-10}}{5^{12}} \times \frac{5^{-12}}{5^{-12}} \\ &= 5^{-22} \end{aligned}$$

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Then, for that y,

$$\begin{aligned}
(a^x)^{y-1} &= (a^x)^y \cdot \frac{1}{a^x} && \text{[recursive definition]} \\
&= (a^x)^y \cdot a^{-x} && \text{[Ex. 1 of Part D]} \\
&= a^{xy} \cdot a^{-x} && \text{[inductive hypothesis]} \\
&= a^{xy-x} && \text{[addition rule]} \\
&= a^{x(y-1)} && \text{[distributive principle]}
\end{aligned}$$

So, for every integer $y \leq 0$, if $(a^x)^y = a^{xy}$ then $(a^x)^{y-1} = a^{x(y-1)}$.

Hence, by principle (I) it follows from (i) and (ii_I) that for every $a \neq 0$, for every integer x, and for every integer $y \geq 0$, $(a^x)^y = a^{xy}$. By principle (II) it follows from (i) and (ii_{II}) that for every $a \neq 0$, for every integer x, and for every $y \leq 0$, $(a^x)^y = a^{xy}$. Therefore, since for every integer y, either $y \leq 0$ or $y \geq 0$, we have proved the first statement in the box on page 2-19.

(2) (a) For all integers $x > 0$ and $y > 0$,

$$(0^x)^y = 0^y = 0 \text{ and } 0^{xy} = 0.$$

(b) For all integers $x > 0$ and $y = 0$,

$$(0^x)^y = 0^0 = 1 \text{ and } 0^{xy} = 0^0 = 1.$$

(c) For every integer $y > 0$,

$$(0^0)^y = 1^y = 1 \text{ and } 0^{0y} = 0^0 = 1.$$

(d) $(0^0)^0 = 1^0 = 1$ and $0^{0 \times 0} = 0^0 = 1$.

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[Stress that division by a number is equivalent to multiplication by the reciprocal of that number.]

Part E.

(1) The property in question is that expressed by:

for every integer x, and for every real number $a \neq 0$,

$$(a^x)^{\cdots} = a^{x \cdot \cdots}.$$

(i) 0 has the property.

$(a^x)^0 = 1$ [recursive definition]

$a^{x \cdot 0} = a^0 = 1$

(ii$_I$) The property is hereditary for integers ≥ 0,

Suppose that, for some integer $y \geq 0$,

$$(a^x)^y = a^{xy}.$$

Then for that y,

$$\begin{aligned}(a^x)^{y+1} &= (a^x)^y \cdot a^x && \text{[recursive definition]}\\ &= a^{xy} \cdot a^x && \text{[inductive hypothesis]}\\ &= a^{xy+x} && \text{[addition rule]}\\ &= a^{x(y+1)}. && \text{[distributive principle]}\end{aligned}$$

So, for every integer $y \geq 0$, if $(a^x)^y = a^{xy}$ then $(a^x)^{y+1} = a^{x(y+1)}$.

(ii$_{II}$) The property is hereditary for integers ≤ 0.

Suppose that, for some integer $y \leq 0$,

$$(a^x)^y = a^{xy}.$$

(continued on T. C. 19C)

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C. (Cont.)

1. 7^{-5}	2. 1	3. 2^{-6}
4. $\left[\frac{1}{2}\right]^{-6}$	5. $\left[\frac{2}{3}\right]^{-5}$	6. $\left[\frac{9}{4}\right]^{9}$
7. 3^{-2}	8. 0	9. 1
10. 6^{-6}	11. 9^{-11}	12. $8^{1} = 8$
13. 7^{20}	14. 3^{4}	15. π^{-8}
16. x^{7}	17. y^{-2}	

Part D.

1. For every real number $a \neq 0$, and for every integer x,

$$a^{x} \cdot a^{-x} = a^{x-x} = a^{0} = 1.$$

Since $a^{x} \cdot a^{-x} = 1$, $a^{x} \neq 0$. So,

$$\frac{a^{x} \cdot a^{-x}}{a^{x}} = \frac{1}{a^{x}}$$

or

$$a^{-x} = \frac{1}{a^{x}} .$$

Alternative Proof:

Since $\frac{a^{x}}{a^{y}} = a^{x} \cdot \frac{1}{a^{y}}$ and, by Exercise 1, $\frac{1}{a^{y}} = a^{-y}$,

$$\frac{a^{x}}{a^{y}} = a^{x} \cdot a^{-y} = a^{x+(-y)} = a^{x-y}.$$

(continued on T. C. 19B)

1. $7^{-2} \times 7^{-3}$
2. $3^{5} \times 3^{-5}$
3. $2^{2} \times 2^{-8}$
4. $\left[\frac{1}{2}\right]^{-3} \times \left[\frac{1}{2}\right]^{-3}$
5. $\left[\frac{2}{3}\right]^{5} \times \left[\frac{2}{3}\right]^{-10}$
6. $\left[\frac{9}{4}\right]^{-6} \times \left[\frac{9}{4}\right]^{15}$
7. $3^{0} \times 3^{-3} \times 3^{1}$
8. $0^{7} \times 0^{3}$
9. $5^{1} \times 5^{-1}$
10. $\frac{6^{3} \cdot 6^{-7}}{6^{2}}$
11. $\frac{9^{-2} \cdot 9^{-5}}{9^{4}}$
12. $\frac{8^{3} \cdot 8^{-7}}{8^{-5}}$
13. $\frac{7^{10}}{7^{-10}}$
14. $\frac{3^{1} \cdot 3^{2} \cdot 3^{3}}{3^{4} \cdot 3^{-2}}$
15. $\frac{\pi^{3} \cdot \pi^{-7}}{\pi^{-3} \cdot \pi^{7}}$
16. $\frac{x^{-2}x^{5}}{x^{3}x^{-7}}$, $[x \neq 0]$
17. $\frac{y^{-5}y^{-3}y^{0}}{y^{-2}y^{-4}}$, $[y \neq 0]$

D. Use the addition rule for exponents to prove each of the following theorems.

1. For every real number $a \neq 0$, and for every integer x,

$$a^{-x} = \frac{1}{a^{x}} \quad \text{and} \quad a^{x} \neq 0.$$

2. For every real number $a \neq 0$, and for all integers x and y,

$$\frac{a^{x}}{a^{y}} = a^{x-y}.$$

E. Use the two principles on page 2-17 to prove the following theorem.

> For every real number $a \neq 0$, and for all integers x and y,
>
> $$(a^{x})^{y} = a^{xy}.$$
>
> For all non-negative integers x and y,
>
> $$(0^{x})^{y} = 0^{xy}.$$

The theorem proved in Part A on page 2-11 can be stated:

For all non-zero real numbers
a and b, and for every integer
$x \geq 0$,

$$(ab)^x = a^x b^x.$$

Combining the theorem with the one just proved yields the first statement in the box on page 2-20. [Note that '$ab \neq 0$' is equivalent to '$a \neq 0$ <u>and</u> $b \neq 0$'. In pointing out this equivalence ask students if they can state a single inequality which is equivalent to '$a \neq 0$ <u>or</u> $b \neq 0$'. One such inequality is '$a^2 + b^2 \neq 0$'.] The second statement in the box follows from:

For every integer $x > 0$, $0^x = 0$,

and:

$$0^0 = 1.$$

Part F.

Property is that expressed by:

for all non-zero real numbers a and b,

$$(ab)^{\cdots} = a^{\cdots}b^{\cdots}.$$

(i) 0 has the property.

$$(ab)^0 = 1$$

$$a^0b^0 = 1 \cdot 1 = 1$$

(ii) The property is hereditary.

Suppose that, for some $x \leq 0$,

$$(ab)^x = a^x b^x .$$

Then, for that x,

$$\begin{aligned}
(ab)^{x-1} &= (ab)^x \cdot \frac{1}{ab} && \text{[recursive definition]} \\
&= (ab)^x \cdot \frac{1}{a} \cdot \frac{1}{b} \\
&= (ab)^x \cdot a^{-1} \cdot b^{-1} \\
&= a^x b^x \cdot a^{-1} \cdot b^{-1} && \text{[inductive hypothesis]} \\
&= a^{x-1} \cdot b^{x-1} && \text{[addition rule]}
\end{aligned}$$

So, for every integer $x \leq 0$, if $(ab)^x = a^x b^x$ then $(ab)^{x-1} = a^{x-1}b^{x-1}$.

Hence, by principle II, the property in question holds for all integers ≤ 0.

(continued on T. C. 20B)

F. Use principle (II) on page 2-17 to prove the following theorem.

For all non-zero real numbers a and b, and for every integer $x \le 0$,

$$(ab)^x = a^x b^x .$$

Combine this theorem with one of the theorems you proved in Part A on page 2-11 to obtain the following theorem.

> For all real numbers a and b such that $ab \ne 0$, and for every integer x,
>
> $$(ab)^x = a^x b^x .$$
>
> For all real numbers a and b such that $ab = 0$, and for every integer $x \ge 0$,
>
> $$(ab)^x = a^x b^x .$$

G. Give non-exponential names for the numbers named by the following expressions.

Sample 1. $\dfrac{(3^{-7} \times 3^5)^2}{(2^4 \times 2^0 \times 2^{-5})^{-3}}$

Solution. $\dfrac{(3^{-7} \times 3^5)^2}{(2^4 \times 2^0 \times 2^{-5})^{-3}}$

$$= \frac{(3^{-2})^2}{(2^{-1})^{-3}}$$

$$= \frac{3^{-4}}{2^3} \qquad \text{[Why?]}$$

$$= 3^{-4} \times \frac{1}{2^3}$$

(continued on next page)

G. (Cont.)

1. $\frac{1}{8}$	2. 1	3. $8^{1/8}$
4. -8	5. $-\frac{1}{8}$	6. 1
7. .000 001	8. .001	9. 1
10. 625	11. $\frac{1}{25}$	12. 25

$$= \frac{1}{3^4} \times \frac{1}{2^3} \qquad \text{[Why?]}$$

$$= \frac{1}{81 \times 8}$$

$$= \frac{1}{648}.$$

Sample 2. $\dfrac{30^{-3} \times 25^{-4} \times 28^5 \times 8^{-3}}{10^{-6} \times 15^{-3} \times 14^3}$

Solution. $\dfrac{30^{-3} \times 25^{-4} \times 28^5 \times 8^{-3}}{10^{-6} \times 15^{-3} \times 14^3}$

$$= \frac{(2 \times 3 \times 5)^{-3} \times (5^2)^{-4} \times (2^2 \times 7)^5 \times (2^3)^{-3}}{(2 \times 5)^{-6} \times (3 \times 5)^{-3} \times (2 \times 7)^3}$$

$$= \frac{(2^{-3} \times 3^{-3} \times 5^{-3}) \times (5^{-8}) \times (2^{10} \times 7^5) \times (2^{-9})}{(2^{-6} \times 5^{-6}) \times (3^{-3} \times 5^{-3}) \times (2^3 \times 7^3)}$$

$$= \frac{2^{-2} \times 3^{-3} \times 5^{-11} \times 7^5}{2^{-3} \times 3^{-3} \times 5^{-9} \times 7^3}$$

$$= 2^{[-2-(-3)]} \times 1 \times 5^{[-11-(-9)]} \times 7^{[5-3]}$$

$$= 2 \times 1 \times 5^{-2} \times 7^2$$

$$= \frac{2 \times 7^2}{5^2}$$

$$= \frac{98}{25}.$$

1. 2^{-3} 2. $2^{-3} \times 2^3$ 3. $2^{-3} + 2^3$

4. $(-2)^3$ 5. $(-2)^{-3}$ 6. $(-2)^3 \times (-2)^{-3}$

7. 10^{-6} 8. $10^3 \times 10^{-6}$ 9. $(4^2 \times 8^{-8})^0$

10. $\dfrac{1}{5^{-4}}$ 11. $\dfrac{5^{20}}{5^{22}}$ 12. $\dfrac{5^{-20}}{5^{-22}}$

(continued on next page)

G. (Cont.)

13. 4	14. $-\frac{1}{27}$	15. $\frac{100}{3}$
16. $\frac{1}{6}$	17. $\frac{1}{720}$	
18. 8	19. $\frac{3}{5}$	

H. 1. x^{m+p-q} 2. a^{-a-b-c}

3. $a^{k+m}b^{-k-m}$

or $\left(\frac{a}{b}\right)^{k+m}$

4. $6^{2p}\left(\frac{a}{b}\right)^{3p}$

5. $3^{-d}5^{d}7^{3d}s^{23d}t^{-9d}$

6. $2^{4}x^{3}b^{-2}d^{2}$

13. $\dfrac{(-2)^{-3}}{(-2)^{-5}}$

14. $\dfrac{(-3)^{2}}{(-3)^{5}}$

15. $\dfrac{(-5)^{2}}{2^{-1} + 4^{-1}}$

16. $\dfrac{(\frac{1}{2})^{-3}(\frac{1}{3})^{-3}}{(\frac{1}{3})^{-4}(\frac{1}{2})^{-4}}$

17. $\dfrac{(\frac{2}{5})^{-2}(\frac{1}{3})^{4}}{(\frac{1}{5})^{-3}(\frac{2}{3})^{2}}$

18. $\dfrac{24^{-3} \times 48^{3}}{36^{-2} \times 6^{4}}$

19. $\dfrac{5^{-3} \times 15^{4} \times 10^{-3}}{(1.5)^{3} \times (-3)^{0} \times 5^{-1}}$

H. Simplify by applying the preceding theorems, stating whatever restrictions must be made in each case so that the simplest expression is equivalent to the given one.

Sample. $\dfrac{(3xy)^{a}(4x^{2}y^{-2})^{-b}}{(5x^{3}y)^{b}(2xy^{3})^{-a}}$

Solution. $\dfrac{(3^{a}x^{a}y^{a})(2^{-2b}x^{-2b}y^{2b})}{(5^{b}x^{3b}y^{b})(2^{-a}x^{-a}y^{-3a})}$

$$= \frac{2^{-2b}3^{a}x^{a-2b}y^{a+2b}}{2^{-a}5^{b}x^{-a+3b}y^{-3a+b}}$$

$$= 2^{a-2b}3^{a}5^{-b}x^{2a-5b}y^{4a+b}$$

Restrictions: $x \neq 0$ and $y \neq 0$ unless $a = 0$ and $b = 0$; domain of 'a' and 'b' is the set of all integers.

1. $x^{m}x^{p}x^{-q}$

2. $a^{-b}a^{-c}a^{-a}$

3. $a^{k}b^{-m}a^{m}b^{-k}$

4. $[ab]^{-p}[6a^{2}b^{-1}]^{2p}$

5. $\dfrac{[3st^{2}]^{-d}[5s^{3}t^{-3}]^{2d}}{[5s^{-3}t^{-2}]^{d}[7s^{5}t^{-1}]^{-3d}}$

6. $\dfrac{36^{2}x^{7}b^{-5}c^{3}d^{2}}{81x^{4}b^{-3}c^{3}}$

(continued on next page)

n $\urcorner(k + m)$

So, for every integer $x \leq 0$, if $a^x > 0$ then $a^{x-1} > 0$. Hence, by principles (I) and (II) the property holds for all integers.

* * *

J. For ease in reading we suggest that decimal numerals for numbers which are less than 1 be written with the digits grouped three to a group. For example the answer to the Sample could be written:

0.000 001.

1.	10	2.	1000	3.	0.001
4.	1	5.	0.1	6.	0.000 01
7.	1,000,000,000	8.	0.0001	9.	100
10.	0.01	11.	0.000 000 1	12.	1,000,000,000,000

K.

1.	10^4	2.	10^6	3.	10^{10}
4.	10^{-3}	5.	10^{-2}	6.	10^{-9}
7.	10^{-1}	8.	10^{-4}	9.	10^5

$m \rceil (k+m)$

H. (Cont.)

7. $t^{-2ps + s - p + stp^2}$ 8. 1

9. $a^{r+s} b^{q+s} c^{r+q}$

Part I.

Property is that expressed by:

for every real $a > 0$,

$a^{\cdots} > 0$.

(i) 0 has the property.

$a^0 = 1 > 0$

(ii_I) The property is hereditary for integers ≥ 0.

Suppose that, for some integer $x \geq 0$, $a^x > 0$.

Then, for that integer x,

$$a^{x+1} = a^x \cdot a \quad \text{[recursive definition]}$$

$$> 0 \cdot a \quad \text{[inductive hypothesis and the hypothesis that } a > 0\text{]}$$

$$= 0.$$

So, for every integer $x \geq 0$, if $a^x > 0$ then $a^{x+1} > 0$.

(ii_{II}) The property is hereditary for integers ≤ 0.

Suppose that, for some integer $x \leq 0$.

$a^x > 0$.

Then, for that integer x,

$$a^{x-1} = a^x \cdot \frac{1}{a} \quad \text{[recursive definition]}$$

$$> 0 \cdot \frac{1}{a} \quad \text{[inductive hypothesis and '}\frac{1}{a} > 0\text{'.]}$$

$$= 0.$$

(continued on T. C. 23B)

7. $$\frac{[t^{s-r}t^{r-p}]^s \times \left[\frac{t^s}{t^p}\right]}{[t^s t^p]^s \div [t^{stp}]^p}$$

8. $$\left[\frac{x^k}{x^j}\right]^{(j+k)}\left[\frac{x^j}{x^m}\right]^{(m+j)}\left[\frac{x^m}{x^k}\right]^{(k+m)}$$

9. $$\frac{[bc]^{rs}[ca]^{sq}[ab]^{qr}}{[b^{r-1}c^{s-1}]^q[c^{s-1}a^{q-1}]^r[a^{q-1}b^{r-1}]^s}$$

I. Use principles (I) and (II) on page 2-17 to prove the following theorem.

For every real $a > 0$, and every integer x, $a^x > 0$.

J. Write a non-exponential name for each of the powers listed in the following exercises. For those powers which are less than 1, give a decimal numeral rather than a common fraction.

Sample. 10^{-6}

Solution. $$10^{-6} = \frac{1}{10^6} = \frac{1}{1,000,000} = 0.000001.$$

1. 10^1	2. 10^3	3. 10^{-3}
4. 10^0	5. 10^{-1}	6. 10^{-5}
7. 10^9	8. 10^{-4}	9. 10^2
10. 10^{-2}	11. 10^{-7}	12. 10^{12}

K. Write an exponential name for each of the numbers listed below.

1. 10,000	2. 1000,000	3. 10,000,000,000
4. 0.001	5. 0.01	6. 0.000000001
7. 0.1	8. $\frac{1}{10000}$	9. $\frac{1}{0.00001}$

I. 1. 2000 2. 5,300,000

3. 0.8 4. 0.03

5. 0.000 71 6. 41.6

7. 0.416 8. 59,000

9. 0.000 59 10. 2.63

11. 0.000 002 63 12. 538

13. 11,400 14. 0.286 3

15. 6,210,000 16. 0.000 049 8

L. Each of the following expressions contains a decimal numeral, a '$\times$', and an exponential for a power of 10. For each expression write an equivalent expression which does not contain an exponential.

Sample. 3.7×10^{-3}

Solution. 3.7×10^{-3}

$= 3.7 \times 0.001$

$= 0.0037.$

1. 2×10^{3}
2. 53×10^{5}
3. 8×10^{-1}
4. 3×10^{-2}
5. 71×10^{-5}
6. 4.16×10^{1}
7. 4.16×10^{-1}
8. 5.9×10^{4}
9. 5.9×10^{-4}
10. 0.00263×10^{3}
11. 0.00263×10^{-3}
12. 538×10^{0}
13. 1.14×10^{4}
14. 2.863×10^{-1}
15. 6.21×10^{6}
16. 4.98×10^{-5}

2.04 Scientific notation.--Often scientists and engineers need to refer to very large or very small numbers. For example, the mass of the earth is about 6,595,000,000,000,000,000,000 tons, the average distance from the earth to the sun is about 93,000,000 miles, and the mass of an electron is about 0.00000000000000000000000000000201 pounds. As you can see, it is very inconvenient to write so many '0's in a numeral. If you were going to copy one of these numerals, you would probably count the number of '0's first. Therefore, it might be easier to write the numeral by telling how many '0's there are in it. For example you might state that the mass of the earth in tons is given by the numeral '6,595' followed by 18 '0's. As you have learned from the preceding exercises, a shorter way of writing

It has been suggested that we discuss a notation, called 'floating point notation' which is closely related to scientific notation. It is much used in the computing field. A floating point notation for a number consists of:

a decimal numeral (not necessarily for a number between 1 and 10),
a times sign, and
an exponential for a power of ten.

Thus, the scientific notation for a number is <u>one</u> of the floating point notations for the number. Scientific notation is especially convenient when one is interested in comparing numbers. Other floating point notations are particularly useful in addition problems. For example:

$$\begin{aligned} & 0.25 \times 10^3 + 0.52 \times 10^5 \\ = \ & 0.25 \times 10^3 + 52 \times 10^3 \\ = \ & 52.25 \times 10^3 \end{aligned}$$

You may wish to mention this generalization of scientific notation to your class.

A. 1. 540 | 2. 63.28
3. .000 900 1 | 4. 603,800
5. 92,900,000 | 6. 6,595,000,000,000,000,000,000
7. 0.000 01 | 8. 0.331 4
9. 1.41 | 10. 70,000
11. 0.08 | 12. 1,318.9
13. 400,100,000 | 14. 0.000 000 040 01
15. 2,010,000,000,000,000,000,000,000,000,000
16. 0.000 000 000 832 94

this numeral is:

$$6{,}595 \times 10^{18}.$$

In order to be able to compare numbers rapidly and to compute with them easily, scientists and engineers have adopted a standard notation for such numbers. The number is expressed by a numeral consisting of the parts:

> a decimal numeral for a number between 1 and 10,
> a times sign, '$\times$' or '$\cdot$', and
> an exponential for a power of ten.

Thus, the mass of the earth is given in <u>scientific</u> <u>notation</u> by:

$$6.595 \times 10^{21} \text{ tons},$$

the average distance from earth to sun by:

$$9.3 \times 10^{7} \text{ miles},$$

and the mass of an electron by:

$$2.01 \times 10^{-30} \text{ pounds}.$$

EXERCISES

<u>A</u>. Give a decimal numeral for each of the numbers which is named below in scientific notation.

1. 5.4×10^{2}
2. 6.328×10^{1}
3. 9.001×10^{-4}
4. 6.038×10^{5}
5. $9.29 \cdot 10^{7}$
6. $6.595 \cdot 10^{21}$
7. $1.00 \cdot 10^{-5}$
8. $3.314 \cdot 10^{-1}$
9. 1.41×10^{0}
10. 7×10^{4}
11. 8×10^{-2}
12. 1.3189×10^{3}
13. $4.001 \cdot 10^{8}$
14. $4.001 \cdot 10^{-8}$
15. 2.01×10^{30}
16. $8.3294 \cdot 10^{-10}$

C. 1. 5.36×10^{2} 2. 1.432×10^{1}

3. 1.3×10^{-3} 4. 5.3×10^{-1}

5. 6.918×10^{6} 6. 4.918×10^{9}

7. 8.1204×10^{-6} 8. 4.8192×10^{3}

9. 9.0×10^{-5} 10. 9.832×10^{2}

11. 8.92×10^{3} 12. 4.31×10^{1}

13. 6.32×10^{3} 14. 6.321×10^{4}

15. 4.91×10^{-2} 16. 5.2×10^{4}

17. 3.814×10^{4} 18. 4.208×10^{2}

19. 6.921142×10^{5} 20. 5.0×10^{-7}

21. 5.8×10^{1} 22. 3.21×10^{5}

23. 3.21×10^{-7} 24. 8.621×10^{3}

25. 5.01×10^{0} 26. 1.001×10^{-6}

B. State a rule for determining whether a number named by a numeral in scientific notation is greater than, equal to, or smaller than 1. Which of the numbers listed in Part A is the largest? The smallest?

C. Write a numeral in scientific notation for each of the numbers listed below.

1. 536	2. 14.32
3. 0.0013	4. 0.53
5. 6,918,000	6. 4,918,000,000
7. 0.0000081204	8. 4819.2
9. 0.00009	10. 983.2

Sample 1. 3815×10^4

Solution. This numeral is not in scientific notation because 3815 is greater than 10. Since $3815 = 3.815 \times 10^3$,

$$\begin{aligned} 3815 \times 10^4 &= (3.815 \times 10^3) \times 10^4 \\ &= 3.815 \times (10^3 \times 10^4) \\ &= 3.815 \times 10^7 \end{aligned}$$

11. 89.2×10^2	12. 431×10^{-1}
13. 0.632×10^4	14. 63.21×10^3
15. 49.1×10^{-3}	16. 0.00052×10^8
17. 381.4×10^2	18. 42.08×10^1
19. $6,921,142 \times 10^{-1}$	20. 0.00005×10^{-2}
21. 0.0058×10^4	22. 0.321×10^6
23. 0.321×10^{-6}	24. 8.621×10^3
25. 0.0501×10^2	26. 0.0001001×10^{-2}

(continued on next page)

C. (Cont.)

In connection with the multiplication exercises in Part C you may want to ask the following question.

Given two numbers a and b such that $1 \leq a < 10$ and $1 \leq b < 10$. Under what condition is $ab < 10$? > 10? > 100?

27. 8.4×10^{1}	28. 3.8×10^{9}
29. 4.41×10^{-7}	30. 2.624×10^{3}
31. 2.624×10^{7}	32. 2.624×10^{3}
33. 2.624×10^{12}	34. 3.61221×10^{5}
35. 3.61221×10^{-8}	36. 3.61221×10^{6}

D.

1. 2.0×10^{-8}	2. 2.0×10^{9}
3. 6.2×10^{-4}	4. 1.1×10^{-1}
5. 6.8×10^{8}	6. 6.8×10^{-8}

Sample 2. 0.0000076×42000

Solution. 0.0000076×42000

$$= (7.6 \times 10^{-6}) \times (4.2 \times 10^{4})$$

$$= (7.6 \times 4.2) \times (10^{-6} \times 10^{4})$$

$$= 31.92 \times 10^{-2}$$

$$= 3.192 \times 10^{-1}$$

27. 4000×0.021
28. $3,800,000 \times 1000$
29. 0.00021×0.0021
30. 41×64
31. $410 \times 64,000$
32. $0.00041 \times 6,400,000$
33. $4.1 \times 10^{7} \times 64,000$
34. 501×721
35. $0.00501 \times .00000721$
36. $5,010,000 \times 0.721$

D. Simplify by dividing. Give answers in scientific notation.

Sample. $\frac{0.0000163}{0.00000074}$

Solution. $\frac{0.0000163}{0.00000074} = \frac{1.63 \times 10^{-5}}{7.4 \times 10^{-7}}$

$$= \frac{1.63}{7.4} \times \frac{10^{-5}}{10^{-7}}$$

$$\stackrel{a}{=} .216 \times 10^{2}$$

$$\stackrel{a}{=} .22 \times 10^{2}$$

$$= 2.2 \times 10.$$

1. $\frac{0.00012}{6000}$
2. $\frac{128000}{0.000064}$
3. $\frac{0.032}{52}$
4. $\frac{0.0007159}{0.00631}$
5. $\frac{4281000}{0.00631}$
6. $\frac{0.0004281}{6310}$

(continued on next page)

D. (Cont.)

7. 6.8×10^{1}

8. 6.8×10^{-3}

9. 6.8×10^{1}

10. 6.8×10^{2}

E. 1. 10×10^{9} or 10^{10}

2. 2×10^{1}

3. 2×10^{1}

4. 3×10^{-2}

5. 1×10^{-3}

6. 5×10^{5}

7. 3.6×10^{-1}

8. $\frac{1}{2}$

9. 5×10^{3}

10. 8×10^{4}

11. 2×10^{-6}

12. 1×10^{-7}

13. 4×10^{8}

14. 1.2×10^{11}

15. 3×10^{-9}

16. 3×10^{4}

17. 10^{6}

18. 10^{-7}

7. $\frac{4.281}{0.0631}$

8. $\frac{4.281}{631}$

9. $\frac{0.000000004281}{0.000000000631}$

10. $\frac{4281065943}{6310000}$

E. Estimate each number listed below.

Sample. $\frac{48000 \times 0.0201}{0.00032}$

Solution. $$\frac{48000 \times 0.0201}{0.00032}$$

$$= \frac{(4.8 \times 10^4) \times (2.01 \times 10^{-2})}{3.2 \times 10^{-4}}$$

$$= \frac{4.8 \times 2.01}{3.2} \times \frac{10^4 \times 10^{-2}}{10^{-4}}$$

$$= \frac{4.8 \times 2.01}{3.2} \times 10^6$$

$$\stackrel{a}{=} \frac{5 \times 2}{3} \times 10^6$$

$$\stackrel{a}{=} 3 \times 10^6 .$$

1. 21000×522000

2. $\frac{391 \times 285}{5850}$

3. $\frac{4900}{240}$

4. $0.0024 \times 0.004 \times 3100$

5. $\frac{140}{721 \times 192}$

6. 168×3100

7. 450×0.00082

8. $\frac{1014}{987 \times 2}$

9. $(3.14)(42)(42)$

10. $\frac{(426)(426)}{2}$

11. $(0.0015)(0.0015)$

12. $\frac{(0.0017)(0.0017)}{39.1}$

13. $(21,000)^2$

14. $(4,800)^3$

15. $(0.0201)^5$

16. $(3.14)(42.8)^2(63.1)$

17. $\left[\frac{4}{3}\right](3.14)(58.1)^3$

18. $\left[\frac{1}{3}\right](3.14)(0.0016)^2(0.04)$

(continued on next page)

E. (Cont.)

19. 8×10^2

20. 3×10^2

21. 6×10^2

22. 1.7×10^1

23. 5×10^{-16}

24. 10^{-8}

Section 2.05 contains a careful development of work with radical expressions. This work underlies work with powers for which the exponents are rational numbers. If students develop a sound grasp of radicals, they will have no trouble with rational number exponents.

19. $\dfrac{(1900)^2(0.004)}{19.2}$ 20. $\dfrac{638 \times 492}{891}$

21. $\dfrac{422 \times 963 \times 0.0014}{492 \times 0.0019}$ 22. $\dfrac{(13.1)^2(19.8)^2}{4314}$

23. $\dfrac{(0.00018)^2(0.000802)^2}{48}$ 24. $\dfrac{(0.059)^3}{(11.5)^4}$

2.05 Roots of non-negative real numbers. -- Before explaining powers with real rational exponents, for example,

$$2^{\frac{1}{2}} \qquad (3.2)^{-\frac{5}{3}} \qquad \pi^{\frac{10}{17}}$$

you must learn a little more about powers with integral exponents. In the exercises which follow we shall learn more about such powers by considering the locus of the equation '$y = x^n$' for positive integral values of 'n'.

EXERCISES

A. Draw the locus of each of the following equations.

Sample. $y = x^3$; values of 'x' from -2.5 to +2.5.

Solution. Some of the points in the locus are graphs of the ordered pairs listed below.

(-2.5, -15.625) (2.5, 15.625)

(-2, -8) (2, 8)

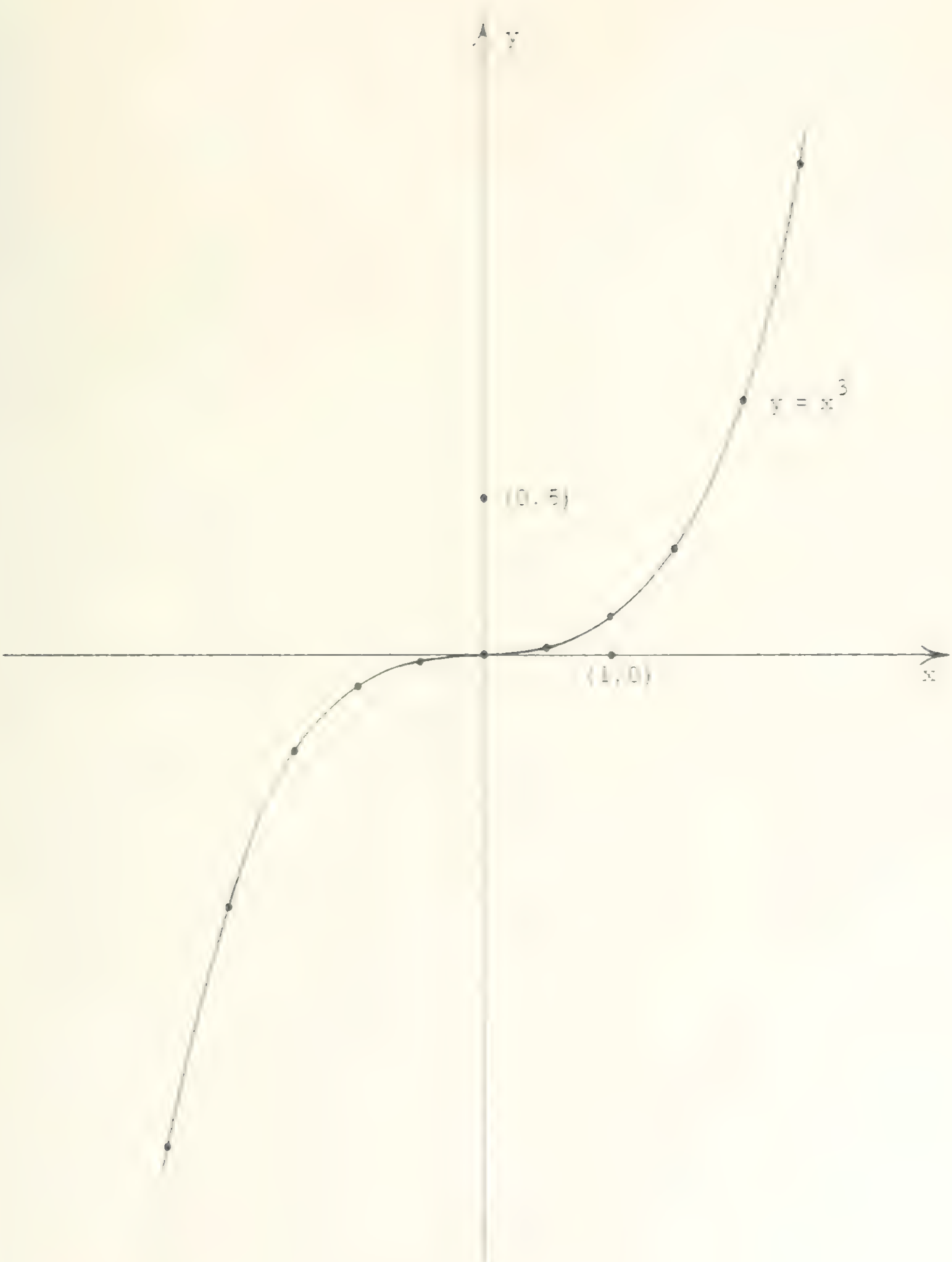
(-1.5, -3.375) (1.5, 3.375)

(-1, -1) (1, 1)

(-0.5, -0.125) (0.5, 0.125)

(0, 0)

If we plot these points and draw a smooth curve through them, we get the following figure.


y
y = x^3
(0, 5)
(1, 0)
x

Part B.

The loci of '$y = x^2$' and '$y = x^4$' are symmetric with respect to the y-axis. The ordinates for the loci are non-negative; the abscissas include all real numbers (positive, 0, or negative). The loci of '$y = x^3$', '$y = x^5$', and '$y = x^9$' are symmetric with respect to the origin. Positive ordinates correspond to positive abscissas; a 0 ordinate corresponds to a 0 abscissa; negative ordinates correspond to negative abscissas.

* * *

Part C.

Here we prove that loci of the form of the locus of '$y = x^3$' are symmetric with respect to the origin.

1. $y = x^2$; $-4 \leq x \leq 4$ 2. $y = x^{[illegible]}$; $-2 \leq x \leq 2$

3. $y = x^5$; $-1.5 \leq x \leq 1.5$ 4. $y = x^{[illegible]}$; $-1 \leq x \leq 1$

B. Study the loci you drew in Part A. Tell how the forms of those loci which correspond to even exponents (Exercises 1 and 2) differ from those which correspond to odd exponents (Exercises 3, 4, and Sample).

C. It is the case that for an odd exponent n, the point (x, y) belongs to the locus of '$y = x^n$' if and only if the point (-x, -y) does.

We can prove this by using mathematical induction. The set in question is the set of all odd integers > 0. Let us agree that for every n in this set, n's follower is n + 2. The principle of mathematical induction which we shall use is the following:

> Every property of odd integers > 0
> which
>
> (a) holds for 1,
>
> and is such that
>
> (b) for every odd integer k, if it holds for k, then it holds for k + 2
>
> holds for every odd integer > 0.

Since what we want to prove is that, for every odd integer n > 0, and for every (x, y),

(x, y) satisfies '$y = x^n$' if and only if (-x, -y) satisfies '$y = x^n$',

the property of odd integers > 0 we are concerned with is expressed by:

for every real number x, $(-x)^n = -x^n$.

The justification for the equation in line 8 is the double application of the recursive definition. Similarly, for the replacement of '$(-x)^2$' by 'x^2'.

* * *

The theorem called for in lines 15-17 is: For every even integer $n > 0$, and for every (x, y),

$$(x, y) \text{ satisfies } `y = x^n\text{'}$$

if and only if

$$(-x, y) \text{ satisfies } `y = x^n\text{'}.$$

The proof of this theorem by mathematical induction is very similar to the proof given on page 2-32 except that the first step of the proof involves demonstrating that 2 has the property. Note that since we are dealing with the set of even integers, k's follower is $k + 2$.

The following is a proof which does not use mathematical induction.

By the first theorem we know that for every even integer $n \geq 2$, and for every (x, y),

$$y = x^{n-1}$$

if and only if

$$y = -(-x)^{n-1},$$

that is,

$$-x^{n-1} = (-x)^{n-1}.$$

But, for every x,

if $\quad -x^{n-1} = (-x)^{n-1}$

then $\quad [-x^{n-1}](-x) = [(-x)^{n-1}](-x)$

or $\quad x^n = (-x)^n.$

Proof:

(a) 1 has the property.

$$(-x)^1 = -x = -x^1.$$

(b) The property is hereditary.

Suppose, for a given odd integer k, that

$$(-x)^k = -x^k.$$

Then, for this k,

$$\begin{aligned}(-x)^{k+2} &= (-x)^k \cdot (-x)^2 \\ &= (-x)^k \cdot x^2 \\ &= -x^k \cdot x^2 \\ &= -(x^k \cdot x^2) \\ &= -x^{k+2}.\end{aligned}$$

Hence, by the principle of mathematical induction stated above, the property holds for every odd integer > 0.

As an exercise you should state and prove by mathematical induction a corresponding theorem about the locus of '$y = x^n$' for every even integer $n > 0$.

[Note: Can you prove this second theorem without using mathematical induction but, instead, by using the first theorem and the fact that, for every integer $n > 0$, if n is even then $n - 1$ is odd?]

D. You proved in Part I on page 2-23 that for every real number $x > 0$, and for every integer $n > 0$, $x^n > 0$. In connection with loci of equations of the form '$y = x^n$' we need a more general theorem.

> For all real numbers x_1 and x_2,
> and for every integer $n > 0$,
> if $0 \leq x_1 < x_2$
> then $0 \leq (x_1)^n < (x_2)^n$.

[In terms of a locus this theorem tells you that if you pick two points in the locus of '$y = x^n$' and if those points are in the first quadrant, the point which has the smaller first coordinate has the smaller second coordinate. Verify the theorem with reference to the loci you drew in Part A.]

We prove the theorem by using the principle of mathematical induction over the set of integers > 0. The property in question is expressed by:

$$\text{for all real numbers } x_1 \text{ and } x_2,$$
$$\text{if } 0 \leq x_1 < x_2$$
$$\text{then } 0 \leq (x_1)^{\cdots} < (x_2)^{\cdots} .$$

<u>Proof</u>:

(a) <u>1 has the property.</u>

$$\text{If } 0 \leq x_1 < x_2 \text{ then } 0 \leq (x_1)^1 < (x_2)^1.$$

(b) <u>The property is hereditary.</u>

[That is, for all real numbers x_1 and x_2, if

$$\text{if } 0 \leq x_1 < x_2 \text{ then } 0 \leq (x_1)^k < (x_2)^k$$

then

$$\text{if } 0 \leq x_1 < x_2 \text{ then } 0 \leq (x_1)^{k+1} < (x_2)^{k+1}.$$

We consider each of the two parts of the inequality:

$$0 \leq x_1 < x_2 .$$

(i) If $0 \leq x_1$ then, by the inductive hypothesis,

$$0 \leq (x_1)^k .$$

So, transforming this last inequality by multiplication, we get:

$$0 \cdot x_1 \leq (x_1)^k \cdot x_1 \qquad [x_1 \geq 0]$$

or:

$$0 \leq (x_1)^{k+1} .$$

$$
\begin{aligned}
x^{k+1} &= x^k \cdot x \\
&\leq x \cdot x && [\text{Inductive hypothesis and `} x \geq 0\text{'.}] \\
&\leq x \cdot 1 && [\text{`} x \leq 1\text{' and `} x \geq 0\text{'.}] \\
&= x.
\end{aligned}
$$

Hence, for every integer $k > 0$, if, for every x such that $0 \leq x \leq 1$, $x^k \leq x$, then, for every x such that $0 \leq x \leq 1$, $x^{k+1} \leq x$.

Consequently, by the principle of mathematical induction for positive integers, the property holds for every positive integer.

Then, for this k, and any $x \geq 1$,

$$\begin{aligned} x^{k+1} &= x^k \cdot x \\ &\geq x \cdot x \qquad \text{[Inductive hypothesis and `} x \geq 0\text{'.]} \\ &\geq x \cdot 1 \qquad \text{[`} x \geq 1\text{' and `} x \geq 0\text{'.]} \\ &= x. \end{aligned}$$

Hence, for every integer $k > 0$, if, for every $x \geq 1$, $x^k \geq x$, then, for every $x \geq 1$, $x^{k+1} \leq x$. Consequently, by the principle of mathematical induction for positive integers, the property holds for every positive integer.

2. Property is that expressed by:

For every real number x such that $0 \leq x \leq 1$,

$$x^{\cdots} \leq x.$$

The theorem is proved by mathematical induction over the set of positive integers.

Proof:

(a) <u>1 has the property</u>.

For every real number x such that $0 \leq x \leq 1$, $x^1 \leq x$.

(b) <u>The property is hereditary</u>.

Suppose, for a given positive integer k, that

for every real number x such that $0 \leq x \leq 1$,

$$x^k \leq x.$$

Then, for this k, and any x such that $0 \leq x \leq 1$,

(continued on T. C. 34C)

D. 1. True. [By the theorem on page 2-32.]

2. False. [By the theorem referred to at the end of Part C, the inequality is equivalent to '$2^{10} < 1.5^{10}$'. By the theorem on page 2-32, $1.5^{10} < 2^{10}$. Since < is asymmetric (Unit 1, T. C. 22M), it is not the case that $2^{10} < 1.5^{10}$.]

3. True. [By the theorem of Part C, the inequality is equivalent to '$-(3^{11}) > -(4^{11})$', and so to '$3^{11} < 4^{11}$'. The last is in consequence of the theorem on page 2-32.]

4. True. [$(-5)^{11} < 0 < (-5)^{10}$]

5. False. [$\frac{1}{2}$ is a counter-example.]

6. True. [Proof-sketch: For every $x \geq 1$, $x^8 \neq 0$. Hence, $x^8 \leq x^{10}$ if and only if $1 \leq x^2$.]

E. 1. Property is that expressed by:

For every real number $x \geq 1$, $x^{\cdots} \geq x$.

The theorem is proved by mathematical induction over the set of positive integers.

Proof:

(a) 1 has the property.

For every $x \geq 1$, $x^1 \geq x$. [Inductive definition; $\geq$ is a reflexive relation (i.e. for every x, $x \geq x$; 'irreflexive', Unit 1, T. C. 22M).]

(b) The property is hereditary.

Suppose, for a given positive integer k, that,

for every $x \geq 1$, $x^k \geq x$.

(continued on T. C. 34B)

(ii) If $x_1 < x_2$ then, by the inductive hypothesis,

$$(x_1)^k < (x_2)^k .$$

So,

$$(x_1)^k \cdot x_1 < (x_2)^k \cdot x_2 \qquad [x_1 < x_2]$$

or,

$$(x_1)^{k+1} < (x_2)^{k+1} .$$

Hence, from (i) and (ii), we have that,

if $0 \leq x_1 < x_2$ then $0 \leq (x_1)^{k+1} < (x_2)^{k+1}$.

Therefore, from (a) and (b) and the principle of mathematical induction, the boxed theorem on page 2-32 is proved.

Of each of the following statements decide whether it is true or false, appealing where possible to the foregoing theorems.

1. $2^{17} < 2.0001^{17}$
2. $(-2)^{10} < (1.5)^{10}$
3. $(-3)^{11} > (-4)^{11}$
4. $(-5)^{11} < (-5)^{10}$
5. For every real number $x \geq 0$, $x^8 \leq x^{10}$.
6. For every real number $x \geq 1$, $x^8 \leq x^{10}$.

E. Prove by mathematical induction:

1. For every real number $x \geq 1$, and for every integer $n > 0$,

$$x^n \geq x .$$

2. For every real number x such that $0 \leq x \leq 1$, and for every integer $n > 0$,

$$x^n \leq x .$$

F. We state now an important theorem about real numbers. You were probably aware of this theorem when you drew the loci of equations of the form '$y = x^n$'. In geometric terms, this theorem asserts that every line which is parallel to the x-axis but is not below it intersects in a single point that part of the locus of '$y = x^n$' which is either on the y-axis or to the right of it.

> For every real number $a \geq 0$, and for every integer $n > 0$, there is a unique real number x such that
>
> $x \geq 0$ and $x^n = a$.

The theorem can be divided into two parts:

(i) For every real number $a \geq 0$, and for every integer $n > 0$, there is at least one real number x such that $x \geq 0$ and $x^n = a$. ["There are no holes (and no highest point) in the locus of '$y = x^n$'."]

(ii) For every real number $a \geq 0$, and for every integer $n > 0$, there is at most one real number x such that $x \geq 0$ and $x^n = a$. ["The locus of '$y = x^n$' does not have ripples in the first quadrant."]

The proof of (i) depends on three facts:

(1) For every integer $n > 0$, $0^n = 0$.

(2) For every real number $x \geq 1$, and for every integer $n > 0$, $x^n \geq x$.

(3) For every integer $n > 0$, the locus of '$y = x^n$' is "smooth".

Fact (1), which follows from the definition on page 2-15, assures us that in case $a = 0$, there is a number 0 such that $0^n = 0$.

Fact (2), which you established in Exercise 1 of Part E on page 2-34, assures us that there is no highest point in the locus of '$y = x^n$'.

The notion of a smooth curve will be made more nearly precise in section 2.07.

* * *

Students should be encouraged to say, for example, 'the principal 4th root of 16' instead of 'the 4th root of 16'.

* * *

Fact (3) cannot be established at this time because to do so would require a definition of 'smooth curve', a definition which we are not ready to state.

You will learn in a later course how to establish fact (3) and how to use the three facts to prove part (i) of the theorem. So, we shall accept (i) at this time.

The proof of part (ii) depends upon the boxed theorem in Part D on page 2-32. We leave the proof of (ii) as an exercise for you. [Hint: Suppose that there are two non-negative real numbers (x_1 and x_2) such that $(x_1)^n = a$ and $(x_2)^n = a$. Use the theorem in Part D to show that this leads to a contradiction.]

G. The "uniqueness theorem" stated in the box of Part F permits you to conclude, for example, that the equation:

$$x^5 = 17$$

has precisely one non-negative real root. This means that it is proper to speak of the real number $x \geq 0$ such that $x^5 = 17$. We give a shorter name to this real number:

the principal 5th root of 17

and a still shorter name:

$$\sqrt[5]{17}\ .$$

More generally, for every real number $a \geq 0$, and for every integer $n > 0$,

the real number $x \geq 0$ such that $x^n = a$

is

the principal nth root of a

or

$$\sqrt[n]{a}\ .$$

[Note 1: '$\sqrt[2]{a}$' is further abbreviated '$\sqrt{a}$'. Note 2: $\sqrt[1]{a} = a$.]

Evidently, for every real number $a \geq 0$, and for every integer $n > 0$,

$$\sqrt[n]{a} \geq 0 \quad \text{and} \quad (\sqrt[n]{a})^n = a\ .$$

G. 1. 2   2. 3

3. 1   4. $\sqrt[5]{7}$

5. None   6. None

7. $\sqrt[100]{1.5}$ 8. $\sqrt[98]{\pi}$

9. $\frac{1}{2}$ 10. None

[' $\sqrt[3]{8}$ ' is an acceptable answer to Exercise 1, but students should recognize that this can be simplified. The equation of Exercise 5 has a negative real root, -1; that of Exercise 6 has no real root. The specified roots of the equations in Exercises 4, 7, and 8 have no simpler standard names than those listed above.]

H. If $a < 0$ and $n > 0$ are odd, then $\left(-\sqrt[n]{-a}\right)^n = -\left(\sqrt[n]{-a}\right)^n = -(-a) = a$. On the other hand, for every real number b, if $b^n = a < 0$ then, by the boxed theorem on page 2-32, $b < 0$. Hence $-b > 0$ and $(-b)^n = -a$, so $-b = \sqrt[n]{-a}$, and $b = -\sqrt[n]{-a}$.

For every real number b and every even $n > 1$, $b^n \geq 0$. Hence, for every $a < 0$ and every even $n > 1$ there is no real number b such that $b^n = a$.

Sample. Find the non-negative real number which is a root of the equation:

$$x^4 = 16.$$

Solution. The positive real root of the equation is $\sqrt[4]{16}$. A simple name for this root is '2'. [$2 \geq 0$ and $2^4 = 16$.]

Find the non-negative real root of each of the following equations if the equation has one.

1. $x^3 = 8$
2. $x^5 - 243 = 0$
3. $y^4 = 1$
4. $b^5 = 7$
5. $6x^3 = -6$
6. $6z^4 = -6$
7. $a^{100} = 1.5$
8. $r^{98} = \pi$
9. $x^{-2} = 4$
10. $-y^{-3} = \frac{1}{8}$

H. Use the results in Part C and those stated in Part F to show that it is possible, for every real number $a < 0$, and for every odd integer $n > 0$, to find a unique real number $\sqrt[n]{a}$ such that $(\sqrt[n]{a})^n = a$.

Show that this is impossible for each real $a < 0$, and for each even integer $n > 1$.

I. A symbol such as '$\sqrt[3]{7}$' is called a radical. For each of the powers listed below find a radical name.

Sample. $(\sqrt[6]{7})^2$

Solution. $[(\sqrt[6]{7})^2]^3 = (\sqrt[6]{7})^6$ [Why?]

$= 7$ [Why?]

Since $(\sqrt[6]{7})^2 \geq 0$ and its third power is 7, we know that $(\sqrt[6]{7})^2 = \sqrt[3]{7}$.

I. 1. $\sqrt[4]{13}$ 2. $\sqrt{13}$ 3. $\sqrt{9}$ ['3' is not a radical name.]

4. $\sqrt[12]{2}$ 5. $\sqrt[8]{2}$ 6. $\sqrt[6]{2}$

7. $\sqrt[4]{2}$ 8. $\sqrt[3]{2}$ 9. [Nonsense problem.]

* * *

10. Since $\sqrt[qk]{a} \geq 0$, it follows from the boxed theorem on page 2-32 that $\left(\sqrt[qk]{a}\right)^k \geq 0$. Since, also, $\left[\left(\sqrt[qk]{a}\right)^k\right]^q = \left(\sqrt[qk]{a}\right)^{qk} = a$, it follows that $\left(\sqrt[qk]{a}\right)^k = \sqrt[q]{a}$. [Note that to show that $b = \sqrt[q]{a}$ we have, precisely, to show that $b \geq 0$ and $b^q = a$.]

J. $\left[\left(\sqrt[9]{4}\right)^3\right]^9 = 4^3 = \left[\left(\sqrt[18]{4}\right)^{18}\right]^3 = \left[\left(\sqrt[18]{4}\right)^6\right]^9$. Since there is just one principal 9th root of $\left[\left(\sqrt[9]{4}\right)^3\right]^9$, $\left(\sqrt[9]{4}\right)^3 = \left(\sqrt[18]{4}\right)^6$.

1. $(\sqrt[8]{13})^2$
2. $(\sqrt[8]{13})^4$
3. $(\sqrt[6]{9})^3$
4. $(\sqrt[24]{2})^2$
5. $(\sqrt[24]{2})^3$
6. $(\sqrt[24]{2})^4$
7. $(\sqrt[24]{2})^6$
8. $(\sqrt[24]{2})^8$
9. $(\sqrt[24]{-2})^8$

* * *

10. Prove the following theorem.

For every real number $a \geq 0$,
and for all positive integers k and q,

$$\sqrt[q]{a} = (\sqrt[qk]{a})^k .$$

J. Consider the powers $(\sqrt[8]{7})^6$ and $(\sqrt[12]{7})^9$. Do you think that

$$(\sqrt[8]{7})^6 = (\sqrt[12]{7})^9 ?$$

We can prove that this is the case as follows:

$$\begin{aligned}[(\sqrt[8]{7})^6]^8 &= [(\sqrt[8]{7})^8]^6 \\ &= 7^6 \\ &= [(\sqrt[12]{7})^{12}]^6 \\ &= (\sqrt[12]{7})^{72} \\ &= [(\sqrt[12]{7})^9]^8\end{aligned}$$

Since there is just one principal 8th root of $[(\sqrt[8]{7})^6]^8$, and since $[(\sqrt[8]{7})^6]^8 = [(\sqrt[12]{7})^9]^8$, we know that

$$(\sqrt[8]{7})^6 = (\sqrt[12]{7})^9.$$

Similarly, we could show that $(\sqrt[9]{4})^3 = (\sqrt[18]{4})^6$, that $(\sqrt[7]{5})^2 = (\sqrt[21]{5})^6$, and that $(\sqrt[3]{14})^8 = (\sqrt[12]{14})^{32}$. In fact it appears that:

$\left(\sqrt[8]{9}\right)^3$	$\left(\sqrt[16]{9}\right)^{12}$	$\left(\sqrt[12]{9}\right)^4$	$\left(\sqrt[8]{9}\right)^4$
$\left(\sqrt[16]{9}\right)^6$	$\left(\sqrt[12]{9}\right)^9$	$\left(\sqrt[6]{9}\right)^2$	$\sqrt{9}$
	$\left(\sqrt[4]{9}\right)^3$		$\left(\sqrt[10]{9}\right)^5$
$\left(\sqrt[10]{9}\right)^8$	$\left(\sqrt[12]{9}\right)^{10}$	$\left(\sqrt[9]{9}\right)^6$	
$\left(\sqrt[15]{9}\right)^{12}$	$\left(\sqrt[18]{9}\right)^{15}$	$\left(\sqrt[6]{9}\right)^4$	
$\left(\sqrt[5]{9}\right)^4$	$\left(\sqrt[6]{9}\right)^5$	$\left(\sqrt[3]{9}\right)^2$	
$\left(\sqrt[20]{9}\right)^{16}$	$\left(\sqrt[24]{9}\right)^{20}$	$\left(\sqrt[12]{9}\right)^8$	

Completion of proof of boxed theorem:

[Reason for first step is Exercise 10, Part I; reason for third step is hypothesis: $\frac{p}{q} = \frac{r}{s}$.]

$$\left(\sqrt[qs]{a}\right)^{qr} = \left[\left(\sqrt[sq]{a}\right)^{q}\right]^{r}$$

$$= \left(\sqrt[s]{a}\right)^{r}.$$

Proof of supplementary statement concerning powers of 0:

Since $\frac{p}{q} = \frac{r}{s}$, $q > 0$, and $s > 0$, and since $p \geq 0$, either p and r are both positive or both 0. But

$$\sqrt[q]{0} = 0, \text{ so } \left(\sqrt[q]{0}\right)^{p} = 0 \text{ if } p > 0, \text{ and } = 1 \text{ if } p = 0,$$

and

$$\sqrt[s]{0} = 0, \text{ so } \left(\sqrt[s]{0}\right)^{r} = 0 \text{ if } r > 0, \text{ and } = 1 \text{ if } r = 0.$$

In either case, $\left(\sqrt[q]{a}\right)^{p} = \left(\sqrt[s]{a}\right)^{r}$.

Exercises.

1. (a) 4 (b) 1
   (c) 1 (d) 4
   (e) 8 (f) 10

2.

$\left(\sqrt[24]{9}\right)^{9}$	$\left(\sqrt[20]{9}\right)^{15}$	$\left(\sqrt[9]{9}\right)^{3}$	$\left(\sqrt[4]{9}\right)^{2}$
$\left(\sqrt[32]{9}\right)^{12}$	$\left(\sqrt[8]{9}\right)^{6}$	$\left(\sqrt[3]{9}\right)$	$\left(\sqrt[6]{9}\right)^{3}$

(continued on T. C. 39B)

> For every real number $a > 0$, and all integers p, q, r, and s such that $q > 0$ and $s > 0$, and
>
> $$\frac{p}{q} = \frac{r}{s},$$
>
> $$(\sqrt[q]{a})^p = (\sqrt[s]{a})^r;$$
>
> if $a = 0$ and $p \geq 0$ then $(\sqrt[q]{a})^p = (\sqrt[s]{a})^r$.

Use the theorem in Exercise 10 of Part I to prove the boxed statement.

Here are the first three steps of the proof.

$$\begin{aligned}(\sqrt[q]{a})^p &= [(\sqrt[qs]{a})^s]^p &\quad [\text{Why?}]\\ &= [\sqrt[qs]{a}]^{sp} & \\ &= [\sqrt[qs]{a}]^{qr} &\quad [\text{Why?}]\end{aligned}$$

1. Use the theorem you have just proved to solve the following equations.

   (a) $(\sqrt[5]{7})^2 = (\sqrt[10]{7})^x$ (b) $(\sqrt[6]{9})^3 = (\sqrt{9})^y$

   (c) $(\sqrt[x]{4})^7 = (\sqrt[3]{4})^{21}$ (d) $(\sqrt[12]{9})^y = (\sqrt[36]{9})^{12}$

   (e) $(\sqrt{3})^5 = (\sqrt[x]{3})^{20}$ (f) $(\sqrt[5]{2})^x = (\sqrt[x]{2})^{20}$

2. Partition the set of expressions which follow according to the relation IS A NAME FOR THE SAME NUMBER AS.

$(\sqrt[24]{9})^9$	$(\sqrt[20]{9})^{15}$	$(\sqrt[32]{9})^{12}$	$(\sqrt[9]{9})^3$
$(\sqrt[8]{9})^6$	$\sqrt[3]{9}$	$(\sqrt[4]{9})^2$	$(\sqrt[10]{9})^8$
$(\sqrt[12]{9})^4$	$(\sqrt[16]{9})^{12}$	$(\sqrt[6]{9})^3$	$(\sqrt[12]{9})^{10}$
$(\sqrt[8]{9})^4$	$(\sqrt[12]{9})^9$	$(\sqrt[15]{9})^{12}$	$(\sqrt[18]{9})^{15}$

(continued on next page)

K. 1. $$\left[\left(\sqrt[q]{a}\right)^p\right]^q = \left[\left(\sqrt[q]{a}\right)^q\right]^p$$

$$= a^p$$

Hence, since $\left(\sqrt[q]{a}\right)^p \geq 0$, $\left(\sqrt[q]{a}\right)^p$ is the principal q<u>th</u> root of a^p.

2. $$\left(\sqrt[q]{\sqrt[s]{a}}\right)^{qs} = \left[\left(\sqrt[q]{\sqrt[s]{a}}\right)^q\right]^s$$

$$= \left(\sqrt[s]{a}\right)^s$$

$$= a$$

Hence, since $\sqrt[q]{\sqrt[s]{a}} \geq 0$, $\sqrt[q]{\sqrt[s]{a}}$ is the principal qs<u>th</u> root of a.

$(\sqrt[6]{9})^5$ $(\sqrt[5]{9})^4$ $(\sqrt[9]{9})^6$ $(\sqrt[6]{9})^2$

$(\sqrt[6]{9})^4$ $\sqrt{9}$ $(\sqrt[8]{9})^3$ $(\sqrt[4]{9})^3$

$(\sqrt[3]{9})^2$ $(\sqrt[10]{9})^5$ $(\sqrt[20]{9})^{16}$ $(\sqrt[24]{9})^{20}$

$(\sqrt[16]{9})^6$ $(\sqrt[12]{9})^8$

K. Prove each of the following theorems.

1. For every real number $a > 0$, and for all integers p and q such that $q > 0$,

$$(\sqrt[q]{a})^p = \sqrt[q]{a^p} \;;$$

for all integers p and q such that $p \geq 0$ and $q > 0$,

$$(\sqrt[q]{0})^p = \sqrt[q]{0^p} \,.$$

[Hint: If $X = (\sqrt[q]{a})^p$ then $X \geq 0$ and $X^q = a^p$.]

Here are several instances of this theorem. Make up six more.

$(\sqrt[3]{2})^5 = \sqrt[3]{32}$ $\sqrt[3]{8^9} = (\sqrt[3]{8})^9 = 2^9$

$\sqrt[5]{17^3} = (\sqrt[5]{17})^3$ $\sqrt{4^5} = (\sqrt{4})^5 = 32$

* * *

2. For every real number $a \geq 0$, and for all positive integers q and s,

$$\sqrt[q]{\sqrt[s]{a}} = \sqrt[qs]{a} \,.$$

Here are several instances of this theorem. Make up six more.

$\sqrt[3]{\sqrt[5]{9}} = \sqrt[15]{9}$ $\sqrt[8]{256} = \sqrt[4]{\sqrt{256}} = \sqrt[4]{16} = 2$

$\sqrt[10]{144} = \sqrt[5]{12}$ $\sqrt[12]{81} - \sqrt[3]{3}$

* * *

Proof that if $0 < a < 1$ then $a < \sqrt{a}$:

If $0 < a < 1$ then $a^2 < a$. [Transformation of '$a < 1$' by multiplication, justified by '$0 < a$'.]

If $a^2 < a$ then $\sqrt{a^2} < \sqrt{a}$. [Previous theorem.]

If $\sqrt{a^2} < \sqrt{a}$ then $a < \sqrt{a}$. ['$\sqrt{a^2} = (\sqrt{a})^2$' by Exercise 1, Part K.]

If $0 \leq \sqrt[q]{b} \leq \sqrt[q]{a}$ then $b \leq a$. (Boxed theorem on page 2-32.)

If $b \leq a$ then $a \not< b$. (Asymmetry and irreflexiveness of $<$.)

Hence, if $\sqrt[q]{a} \not< \sqrt[q]{b}$ then $a \not< b$.]

[Students may be interested in the fact that although the converse of a conditional statement is not, in general, equivalent to the statement itself, the statement:

$$\text{if } a < b \text{ then } \sqrt[q]{a} < \sqrt[q]{b}$$

can be used in proving its converse:

$$\text{if } \sqrt[q]{a} < \sqrt[q]{b} \text{ then } a < b.$$

Proof of converse:

If $\sqrt[q]{a} < \sqrt[q]{b}$ then $\sqrt[q]{b} \not< \sqrt[q]{a}$. (Asymmetry of $<$.)

If $\sqrt[q]{b} \not< \sqrt[q]{a}$ then $b \not< a$.

If $b \not< a$ then $a \leq b$. (Connectedness of $<$.)

If $a = b$ then $\sqrt[q]{a} = \sqrt[q]{b}$.

If $\sqrt[q]{a} = \sqrt[q]{b}$ then $\sqrt[q]{a} \not< \sqrt[q]{b}$. (Irreflexiveness of $<$.)

Hence, if $\sqrt[q]{a} < \sqrt[q]{b}$ then $a \neq b$.

Consequently, if $\sqrt[q]{a} < \sqrt[q]{b}$ then $a \leq b$ and $a \neq b$, i.e. $a < b$.

Warning!! Students should be thoroughly convinced that a conditional statement, such as 'if a man has red hair then he has a temper', is not equivalent to its converse, 'if a man has a temper then he has red hair', before taking up the preceding proof.]

(continued on T. C. 41C)

3.
$$\left(\sqrt[q]{a} \cdot \sqrt[q]{b}\right)^q = \left(\sqrt[q]{a}\right)^q\left(\sqrt[q]{b}\right)^q$$
$$= ab$$

Hence, since $\sqrt[q]{a}\,\sqrt[q]{b} \geq 0$, $\sqrt[q]{a} \cdot \sqrt[q]{b}$ is the principal $q\underline{th}$ root of ab.

4.
$$\left(\frac{\sqrt[q]{a}}{\sqrt[q]{b}}\right)^q = \frac{(\sqrt[q]{a})^q}{(\sqrt[q]{b})^q}$$
$$= \frac{a}{b}$$

Hence, since $\dfrac{\sqrt[q]{a}}{\sqrt[q]{b}} \geq 0$, $\dfrac{\sqrt[q]{a}}{\sqrt[q]{b}}$ is the principal $q\underline{th}$ root of $\dfrac{a}{b}$.

5. [Proof is given in hint, but this is, of course, a place at which to remind students that a conditional statement is equivalent to its contrapositive.

Statement: if $a < b$ then $\sqrt[q]{a} < \sqrt[q]{b}$

Contrapositive: if $\sqrt[q]{a} \not< \sqrt[q]{b}$ then $a \not< b$

Proof of contrapositive:

If $\sqrt[q]{a} \not< \sqrt[q]{b}$ then $\sqrt[q]{b} \leq \sqrt[q]{a}$ (connectedness of <, see Unit 1, T. C. 22N), and $0 \leq \sqrt[q]{b}$. (Definition of 'principal root'.)

(continued on T. C. 41B)

3. For all non-negative real numbers a and b, and for every positive integer q,

$$\sqrt[q]{a} \cdot \sqrt[q]{b} = \sqrt[q]{ab}.$$

[Hint: Use the theorem in the box on page 2-20.]
Here are several instances of this theorem. Make up six more.

$\sqrt[3]{2} \cdot \sqrt[3]{5} = \sqrt[3]{10}$ $\sqrt{5} \cdot \sqrt{20} = 10$

$\sqrt[5]{2} \cdot \sqrt[5]{16} = 2$ $\sqrt[3]{54} = \sqrt[3]{2} \cdot \sqrt[3]{27} = 3\sqrt[3]{2}$

* * *

4. For all non-negative real numbers a and b such that b > 0, and for every positive integer q,

$$\frac{\sqrt[q]{a}}{\sqrt[q]{b}} = \sqrt[q]{\frac{a}{b}} .$$

Here are instances. Make up six more.

$\sqrt[5]{\frac{5}{7}} = \frac{\sqrt[5]{5}}{\sqrt[5]{7}}$ $\frac{\sqrt[3]{16}}{\sqrt[3]{2}} = \sqrt[3]{8} = 2$

$\frac{\sqrt[4]{324}}{\sqrt[4]{4}} = \sqrt[4]{81} = 3$ $\frac{\sqrt{4}}{\sqrt{25}} = \sqrt{\frac{4}{25}} = \frac{2}{5}$

* * *

5. For all non-negative real numbers a and b, and for every positive integer q,

$$\text{if } a < b \text{ then } \sqrt[q]{a} < \sqrt[q]{b} .$$

[Hint: Suppose $\sqrt[q]{a} \not< \sqrt[q]{b}$, that is, $\sqrt[q]{a} \geq \sqrt[q]{b}$. Then, by the theorem in the box on page 2-32, $a \geq b$, that is, $a \not< b$.]

Use the theorem in Exercise 5 to prove that, for every real number a such that $0 < a < 1$,

$$a < \sqrt{a} .$$

Students should make certain that the restrictions stated in the heading of each Sample are sufficient to justify the steps.

L. It is often convenient to transform a radical expression into other radical expressions which are equivlent to the given one. Study the samples until you can justify each step.

Sample 1. For all non-negative real numbers a and b,

$$\begin{aligned}\sqrt[3]{54a^5b^7} &= \sqrt[3]{27a^3b^6 \times 2a^2b}\\ &= \sqrt[3]{(3ab^2)^3 \times 2a^2b}\\ &= \sqrt[3]{(3ab^2)^3} \times \sqrt[3]{2a^2b}\\ &= 3ab^2\sqrt[3]{2a^2b}\ .\end{aligned}$$

Sample 2. For every real y, and for every real $x \geq 0$,

$$\begin{aligned}2x\sqrt[4]{3xy^2} &= \sqrt[4]{(2x)^4}\ \sqrt[4]{3xy^2}\\ &= \sqrt[4]{(2x)^4(3xy^2)}\\ &= \sqrt[4]{16x^4(3xy^2)}\\ &= \sqrt[4]{48x^5y^2}\end{aligned}$$

Sample 3. For all real numbers x and y, and for all real numbers u and v such that $uv \neq 0$,

$$\begin{aligned}\sqrt[3]{\frac{3xy}{8u^3v^3}} &= \frac{\sqrt[3]{3xy}}{\sqrt[3]{8u^3v^3}}\\ &= \frac{\sqrt[3]{3xy}}{2uv}\end{aligned}$$

Sample 4. For all positive real numbers x and y,

$$\sqrt{\frac{3}{2xy^3}} = \sqrt{\frac{3}{2xy^3} \cdot \frac{2xy}{2xy}}$$

$$= \sqrt{\frac{6xy}{(2xy^2)^2}}$$

$$= \frac{\sqrt{6xy}}{2xy^2}$$

Sample 5. For all real a, b, and c such that ac $\geq$ 0,

$$\sqrt[12]{9a^2b^4c^{10}} = \sqrt[12]{(3ab^2c^5)^2}$$

$$= \sqrt[6]{\sqrt[2]{(3ab^2c^5)^2}}$$

$$= \sqrt[6]{3ab^2c^5}$$

or

$$\sqrt[12]{9a^2b^4c^{10}} = \sqrt[12]{(3ab^2c^5)^2}$$

$$= \left[\sqrt[12]{3ab^2c^5}\right]^2$$

$$= \sqrt[6]{3ab^2c^5}$$

Sample 6. For all real x and y such that xy $\geq$ 0,

$$\sqrt[3]{2x^2y} \times \sqrt[4]{3xy^3} = \left[\sqrt[12]{2x^2y}\right]^4\left[\sqrt[12]{3xy^3}\right]^3$$

$$= \sqrt[12]{(2x^2y)^4} \times \sqrt[12]{(3xy^3)^3}$$

$$= \sqrt[12]{(2x^2y)^4 \times (3xy^3)^3}$$

$$= \sqrt[12]{(16x^8y^4)(27x^3y^9)}$$

$$= \sqrt[12]{432x^{11}y^{13}}$$

$$= \sqrt[12]{y^{12}} \times \sqrt[12]{432x^{11}y}$$

$$= y\sqrt[12]{432x^{11}y}$$

Sample 7. "Rationalizing the denominator."

$$\frac{17}{7 - 4\sqrt{2}} = \frac{17}{7 - 4\sqrt{2}} \cdot \frac{7 + 4\sqrt{2}}{7 + 4\sqrt{2}}$$

$$= \frac{17(7 + 4\sqrt{2})}{49 - 16(\sqrt{2})^2}$$

$$= \frac{17(7 + 4\sqrt{2})}{17}$$

$$= 7 + 4\sqrt{2}$$

Sample 8. "Rationalizing the numerator."

For every real number $y \geq 0$,

$$\frac{\sqrt{y+2} - \sqrt{y}}{4} = \frac{\sqrt{y+2} - \sqrt{y}}{4} \times \frac{\sqrt{y+2} + \sqrt{y}}{\sqrt{y+2} + \sqrt{y}}$$

$$= \frac{(\sqrt{y+2})^2 - (\sqrt{y})^2}{4(\sqrt{y+2} + \sqrt{y})}$$

$$= \frac{y + 2 - y}{4(\sqrt{y+2} + \sqrt{y})}$$

$$= \frac{2}{4(\sqrt{y+2} + \sqrt{y})}$$

$$= \frac{1}{2(\sqrt{y+2} + \sqrt{y})}$$

ıf

L. (Cont.)

1. $3yk^2m^6\sqrt{2yk}$

2. $3a^2b^2$

3. $\sqrt[4]{16x^6y^3}$

4. $x\sqrt{1 + 3x^2y^2}$

5. $5ab\sqrt{2ab}$

6. $3x\sqrt[3]{x^3 - xy^2}$

7. $\dfrac{\sqrt{35ab}}{7b}$

8. $\dfrac{\sqrt{39xy}}{3xy^2}$

9. $\dfrac{3yz^2}{2a^2b}\sqrt[3]{a^2bx^2yz}$

10. $4(x - y)\sqrt[3]{(x - y)^2}$

11. $a^2b\sqrt[6]{1125ab^2}$

12. $\dfrac{9(3 - \sqrt{5})}{4}$

13. $\dfrac{-14 - 4\sqrt{3}}{37}$

14. $\dfrac{4x - 4x\sqrt{x}}{1 - x}$

15. $\dfrac{34}{6 - \sqrt{2}}$

16. 1

17. $\sqrt{5} - \sqrt{3}$

18. $x - \sqrt{x^2 - 1}$

19. $\sqrt[8]{5x^2y^3zu^4}$

20. $\sqrt[4]{(3abcx)^2}$ and others

21. $x\left(\sqrt{x} - \sqrt[3]{x^2} + \sqrt[4]{x^3}\right)$

Use the methods illustrated in the samples to transform each of the following radical expressions into one or more equivalent expressions.

1. $\sqrt{128y^3k^5m^{12}}$

2. $\sqrt[3]{3a^2b} \times \sqrt[3]{9a^4b^5}$

3. $2x\sqrt[4]{y}\,\sqrt[2]{xy}$

4. $\sqrt{x^2(1 + 3x^2y^2)}$

5. $\sqrt{5ab^2} \times \sqrt{10a^2b}$

6. $3\sqrt[3]{x^6 - x^4y^2}$

7. $\sqrt{\dfrac{5a}{7b}}$

8. $\sqrt{\dfrac{13}{3xy^3}}$

9. $\sqrt[3]{\dfrac{27x^2y^4z^7}{8a^4b^2}}$

10. $\sqrt[3]{64(x - y)^5}$

11. $\sqrt[3]{3a^2b} \times \sqrt{5a^3b^2}$

12. $\dfrac{9}{3 + \sqrt{5}}$

13. $\dfrac{-2}{7 - 2\sqrt{3}}$

14. $\dfrac{4x}{1 - \sqrt{x}}$

15. $\dfrac{6 + \sqrt{2}}{1}$

16. $\dfrac{7 - \sqrt{x}}{7 - \sqrt{x}}$

17. $\dfrac{2}{\sqrt{5} - \sqrt{3}}$

18. $\dfrac{1}{x + \sqrt{x^2 - 1}}$

19. $\sqrt[24]{125x^6y^9z^3u^{12}}$

20. $\sqrt{3abcx}$

21. $\sqrt{x^3} + \sqrt[3]{x^5} - \sqrt[4]{x^7}$

2.06 Real rational numbers as exponents.--We are now ready to explain powers with rational (non-integral) exponents. We want to do this in such a way that our previous theorems concerning exponents will, insofar as possible, continue to hold when the exponents are rational numbers.

With this in mind, how, for example, should we define '$2^{\frac{1}{4}}$'? If the multiplication rule for exponents is to continue to hold, we must define '$2^{\frac{1}{4}}$' so that

$$\left(2^{\frac{1}{4}}\right)^4 = 2^{\frac{1}{4}\times 4} = 2^1 = 2$$

Hence, it is reasonable to assert that $2^{\frac{1}{4}}$ is one of the two real roots of the equation '$x^4 = 2$'. You have learned that this equation has just one real root ≥ 0. It is $\sqrt[4]{2}$. We shall define '$2^{\frac{1}{4}}$' to be a name for $\sqrt[4]{2}$.

For another example, consider '$6^{\frac{3}{5}}$'. If the multiplication rule is to hold,

$$\left(6^{\frac{3}{5}}\right)^5 = 6^{\frac{3}{5}\times 5} = 6^3$$

Hence, we define '$6^{\frac{3}{5}}$' to be a name for $\sqrt[5]{6^3}$. Since $\sqrt[5]{6^3} = \left(\sqrt[5]{6}\right)^3$, then $6^{\frac{3}{5}} = \left(\sqrt[5]{6}\right)^3$. This latter fact is consistent with the definition of '$6^{\frac{1}{5}}$' as a name for $\sqrt[5]{6}$ because

$$6^{\frac{3}{5}} = \left(6^{\frac{1}{5}}\right)^3 = \left(\sqrt[5]{6}\right)^3.$$

Similar arguments could lead us to define '$2^{\frac{3}{4}}$' to be a name for $\left(\sqrt[4]{2}\right)^3$, and '$2^{\frac{6}{8}}$' to be a name for $\left(\sqrt[8]{2}\right)^6$. Since $\frac{3}{4} = \frac{6}{8}$, the definitions of '$2^{\frac{3}{4}}$' and '$2^{\frac{6}{8}}$' must be consistent with the equation:

$$2^{\frac{3}{4}} = 2^{\frac{6}{8}}$$

That is, '$2^{\frac{3}{4}}$' and '$2^{\frac{6}{8}}$' should be names for the same number. By the theorem on page 2-39,

$$\left(\sqrt[4]{2}\right)^3 = \left(\sqrt[8]{2}\right)^6,$$

and so we are assured that '$2^{\frac{3}{4}}$' and '$2^{\frac{6}{8}}$' are names for the same number.

We are now ready to state a general defining principle.

> For every number $a > 0$, and for every rational number x,
>
> $$a^x = \left(\sqrt[q]{a}\right)^p$$
>
> where p and q are any integers such that $q > 0$ and $x = \frac{p}{q}$;
>
> for every positive rational number x,
>
> $$0^x = 0;$$
>
> and
>
> $$0^0 = 1.$$

The theorem stated on page 2-40 shows that for every real number $a > 0$, and for every rational number x, the number a^x is unique, even though there is an unlimited number of choices for p and q. For example,

$$9^{0.8} = 9^{\frac{4}{5}} = 9^{\frac{80}{100}} = \ldots$$

because

$$\left(\sqrt[10]{9}\right)^8 = \left(\sqrt[5]{9}\right)^4 = \left(\sqrt[100]{9}\right)^{80} = \ldots$$

Does $0^{\frac{3}{4}} = 0^{\frac{6}{8}} = 0^{\frac{24}{32}}$? Does $0^{\frac{0}{2}} = 0^{\frac{0}{3}} = 0^{\frac{0}{17}}$?

A. 1. 256	2. 27	3. 8
4. 4	5. 81	6. 16
7. 27	8. 2	9. 8
10. 52	11. 7776	12. 81
13. 100	14. 100	15. 100,000,000
16. 9	17. 9	18. 1024
19. 0.000 32	20. 25	21. 25

[Note, furthermore, that for every real number $a \geq 0$, the foregoing principle coincides with the explanation previously given for powers with real integral exponents. This is because, for every real $a \geq 0$, $\sqrt[1]{a} = a$. (Explain more fully.)]

EXERCISES

A. In each of the powers listed below write a simple name which is neither an exponential nor a radical.

Sample. $49^{\frac{3}{2}}$

Solution. $49^{\frac{3}{2}} = \left(\sqrt[2]{49}\right)^3 = 7^3 = 343$

1. $16^{\frac{4}{2}}$	2. $9^{\frac{3}{2}}$	3. $16^{\frac{3}{4}}$
4. $16^{\frac{2}{4}}$	5. $27^{\frac{4}{3}}$	6. $1024^{0.4}$
7. $243^{0.6}$	8. $1024^{\frac{1}{10}}$	9. $1024^{0.3}$
10. $52^{\frac{2}{2}}$	11. $216^{\frac{5}{3}}$	12. $81^{\frac{3}{3}}$
13. $10000^{\frac{1}{2}}$	14. $1000000^{\frac{2}{6}}$	15. $1000000^{\frac{8}{6}}$
16. $\left(81^{\frac{1}{4}}\right)^2$	17. $\left(81^2\right)^{\frac{1}{4}}$	18. $256^{1.25}$
19. $(0.008)^{1.66\ldots}$	20. $\left(625^{\frac{2}{5}}\right)^{\frac{5}{4}}$	21. $\left(625^{\frac{5}{4}}\right)^{\frac{2}{5}}$

B. Find a rational approximation accurate to two significant digits for each of the following.

Sample. $10^{\frac{1}{4}}$

Solution. $10^{\frac{1}{4}}$ is the positive number whose 4th power is 10. Since 1^4 is 1, 2^4 is 16, and $1 < 10 < 16$, then, by the theorem in Exercise 5 on page 2-41, $1 < 10^{\frac{1}{4}} < 2$.

B. (Cont.)

1. 2.2   2. 3.3   3. 1.3   4. 128

Thus, your first guess at an approximation might be 1.5. We check this guess by calculating:

$$(1.5)^4 = [(1.5)^2]^2 = [2.25]^2 \stackrel{a}{=} 5.3.$$

Our guess was too small. Try 1.8 and calculate again:

$$(1.8)^4 = (3.24)^2 \stackrel{a}{=} 10.2.$$

So, 1.8 is too large. We could try 1.7 but since we are looking for a rational approximation which is accurate to two significant digits let us use 1.75 and thereby be able to tell whether 1.7 or 1.8 is the sought-for approximation [Explain].

$$(1.75)^4 \stackrel{a}{=} 9.36 < 10$$

Therefore, $1.8 \stackrel{a}{=} 10^{\frac{1}{4}}$, correct to two significant digits.

1. $10^{\frac{1}{3}}$ 2. $5^{\frac{3}{4}}$ 3. $2^{\frac{2}{5}}$ 4. $4^{3.5}$

C. Use the principle in the box on page 2-47 to prove each of the following.

Sample. $3^{\frac{2}{3}} \cdot 3^{\frac{4}{5}} = 3^{\frac{22}{15}}$

Solution.
$$3^{\frac{2}{3}} \cdot 3^{\frac{4}{5}} = 3^{\frac{10}{15}} \cdot 3^{\frac{12}{15}}$$
$$= \left(\sqrt[15]{3}\right)^{10} \cdot \left(\sqrt[15]{3}\right)^{12}$$
$$= \left(\sqrt[15]{3}\right)^{22}$$
$$= 3^{\frac{22}{15}}.$$

C. 5. For rational numbers x and y there exist integers p, q, r, s such that q and s are positive, $x = \frac{p}{q}$, and $y = \frac{r}{s}$. For every $a > 0$,

$$a^x = \left(\sqrt[q]{a}\right)^p, \text{ and } a^y = \left(\sqrt[s]{a}\right)^r.$$

Hence,

$$\begin{aligned} a^x \cdot a^y &= \left(\sqrt[q]{a}\right)^p \cdot \left(\sqrt[s]{a}\right)^r \\ &= \left(\sqrt[qs]{a}\right)^{ps} \cdot \left(\sqrt[qs]{a}\right)^{qr} \\ &= \left(\sqrt[qs]{a}\right)^{ps + qr} \end{aligned}$$

Since $ps + qr$ and qs are integers, $qs > 0$, and $x + y = \frac{ps + qr}{qs}$,

$$\left(\sqrt[qs]{a}\right)^{ps + qr} = a^{x + y}.$$

D. 1. $a^x \cdot a^{-x} = a^{x - x} = a^0 = 1$. Therefore $a^x \neq 0$ and $a^{-x} = \frac{1}{a^x}$.

2. $\frac{a^x}{a^y} = a^x \cdot \frac{1}{a^y} = a^x \cdot a^{-y} = a^{x - y}$.

1. $4^{\frac{1}{5}} \cdot 4^{\frac{2}{3}} = 4^{\frac{13}{15}}$

2. $7^{\frac{1}{2}} \cdot 7^{\frac{1}{3}} = 7^{\frac{5}{6}}$

3. $3^{0.5} \cdot 3^{0.6} = 3^{1.1}$

4. $\pi^{\frac{2}{5}} \cdot \pi^{\frac{3}{4}} = \pi^{\frac{23}{20}}$

5. Prove the following theorem.

> For every real number $a > 0$, and for all rational numbers x and y,
>
> $$a^x \cdot a^y = a^{x+y};$$
>
> for all non-negative rationals x and y,
>
> $$0^x \cdot 0^y = 0^{x+y}.$$

D. The theorem in Exercise 5 of Part C has two corollaries which are analogous to those you proved in Part D on page 2-19. Prove each of these corollaries.

1. For every real number $a > 0$, and for every real rational number x,

$$a^{-x} = \frac{1}{a^x} \quad \text{and} \quad a^x \neq 0.$$

2. For every real number $a > 0$, and for all real rational numbers x and y,

$$\frac{a^x}{a^y} = a^{x-y}.$$

E. Simplify each of the following.

Sample. $\dfrac{5^{-\frac{1}{2}} \, 5^{\frac{1}{3}}}{5^{\frac{2}{5}} \, 5^{-\frac{3}{2}}}$

E. 1. $3^{\frac{19}{15}}$ 2. $7^{\frac{11}{30}}$

3. $\pi^{-\frac{59}{120}}$ 4. 1

5. $\dfrac{1}{2^{\frac{2}{3}} + 2^{-\frac{1}{2}}}$ or $\dfrac{2^{\frac{1}{2}}}{2^{\frac{7}{6}} + 1}$ 6. $3^{\frac{2}{3}} + 3^{-\frac{1}{2}}$ or $\dfrac{3^{\frac{7}{6}} + 1}{3^{\frac{1}{2}}}$

F. 1. $\left(a^{\frac{1}{q}}\right)^p = \left[\left(\sqrt[q]{a}\right)^1\right]^p = \left(\sqrt[q]{a}\right)^p = a^{\frac{p}{q}} = \left(\sqrt[q]{a}\right)^p = \sqrt[q]{a^p} = \left(\sqrt[q]{a^p}\right)^1 =$

$\left(a^p\right)^{\frac{1}{q}}$ $(a \neq 0)$ $0^{\frac{1}{q}} = 0.$ [Since $\frac{1}{q} > 0$]. $\left(0^{\frac{1}{q}}\right)^p =$

$0^p = 0$ if $p > 0$, $= 1$ if $p = 0$. $0^{\frac{p}{q}} = 0$ if $p > 0$ [Since $\frac{p}{q} > 0.$],

$= 1$ if $p = 0$ [Since $\frac{p}{q} = 0.$]. $0^p = 0$ if $p > 0$, $= 1$ if $p = 0$.

Hence, $\left(0^q\right)^{\frac{1}{q}} = 0^{\frac{1}{q}} = 0$ if $p > 0$, $= 1^{\frac{1}{q}} = 1$ if $p = 0$.

2. $\left(a^{\frac{1}{q}}\right)^{\frac{1}{s}} = \left[\left(\sqrt[q]{a}\right)^1\right]^{\frac{1}{s}} = \left(\sqrt[s]{\sqrt[q]{a}}\right)^1 = \sqrt[s]{\sqrt[q]{a}} = \sqrt[qs]{a} = \left(\sqrt[qs]{a}\right)^1 = a^{\frac{1}{qs}}$

<u>Solution</u>. $$\frac{5^{-\frac{1}{2}}\,5^{\frac{1}{3}}}{5^{\frac{2}{5}}\,5^{-\frac{3}{2}}} = 5^{\left(-\frac{1}{2}+\frac{1}{3}-\frac{2}{5}+\frac{3}{2}\right)}$$
$$= 5^{\frac{14}{15}}$$

1. $3^{\frac{2}{3}}\,3^{-\frac{1}{5}}\,3^{\frac{4}{5}}$

2. $7^{-\frac{1}{3}}\,7^{\frac{1}{5}}\,7^{\frac{1}{2}}$

3. $\dfrac{\pi^{\frac{1}{2}}\,\pi^{-\frac{2}{3}}}{\pi^{\frac{1}{5}}\,\pi^{\frac{1}{8}}}$

4. $\dfrac{1^{\frac{9}{11}}\,1^{\frac{5}{17}}\,1^{\frac{18}{37}}}{1^{-\frac{2}{5}}\,1^{\frac{7}{8}}\,1^{-\frac{101}{102}}}$

5. $\dfrac{2^{\frac{1}{2}} \times 2^{-\frac{2}{3}}}{2^{\frac{1}{2}} + 2^{-\frac{2}{3}}}$

6. $\dfrac{3^{\frac{1}{2}} + 3^{-\frac{2}{3}}}{3^{\frac{1}{2}} \times 3^{-\frac{2}{3}}}$

<u>F</u>. Use the theorems in Part K on page 2-40 to prove the following theorems.

1. For every real number $a > 0$, and for all integers p and q such that $q > 0$,

$$\left(a^{\frac{1}{q}}\right)^{p} = a^{\frac{p}{q}} = (a^{p})^{\frac{1}{q}};$$

for all integers $p \geq 0$ and $q > 0$,

$$\left(0^{\frac{1}{q}}\right)^{p} = 0^{\frac{p}{q}} = (0^{p})^{\frac{1}{q}}.$$

2. For every real number $a \geq 0$, and for all positive integers q and s,

$$\left(a^{\frac{1}{q}}\right)^{\frac{1}{s}} = a^{\frac{1}{qs}}.$$

$$
\begin{aligned}
(ab)^x &= \left(\sqrt[q]{ab}\right)^p \\
&= \left(\sqrt[q]{a} \cdot \sqrt[q]{b}\right)^p \\
&= \left(\sqrt[q]{a}\right)^p \cdot \left(\sqrt[q]{b}\right)^p \\
&= a^x \cdot b^x .
\end{aligned}
$$

[The additional cases in which $a = 0$ or $b = 0$ are easily treated, for $x \geq 0$. The second part of the boxed theorem is proved by combining these results with the theorem obtained by adding the restriction that $x \geq 0$ to the theorem just proved. The boxed statement is stated as it is in order to illustrate a mode of statement different from that which we have previously used.]

3. $(ab)^{\frac{1}{q}} = \left(\sqrt[q]{ab}\right)^1 = \left(\sqrt[q]{a} \cdot \sqrt[q]{b}\right)^1 = \left(\sqrt[q]{a}\right)^1\left(\sqrt[q]{b}\right)^1 = a^{\frac{1}{q}} \cdot b^{\frac{1}{q}}$.

G. For rational numbers x and y there exist integers p, q, r, s such that q and s are positive, $x = \frac{p}{q}$, and $y = \frac{r}{s}$. For every $a > 0$,

$$
\begin{aligned}
(a^x)^y &= \left[\sqrt[s]{\left(\sqrt[q]{a}\right)^p}\right]^r \\
&= \left[\left(\sqrt[s]{\sqrt[q]{a}}\right)^p\right]^r \\
&= \left[\left(\sqrt[sq]{a}\right)^p\right]^r \\
&= \left(\sqrt[qs]{a}\right)^{pr}
\end{aligned}
$$

Since pr and qs are integers, $qs > 0$, and $xy = \frac{pr}{qs}$,

$$\left(\sqrt[qs]{a}\right)^{pr} = a^{xy}.$$

If $x > 0$ and $y > 0$, then $(0^x)^y = (0)^y = 0 = 0^{xy}$. If $x > 0$ then $(0^x)^0 = 0^0 = 1 = 0^{x0}$. If $y \geq 0$ then $(0^0)^y = 1^y = 1 = 0^{0 \cdot y}$.

H. For every rational number x there exist integers p and q such that $q > 0$, and $x = \frac{p}{q}$. For $a > 0$ and $b > 0$

(continued on T. C. 52B)

3. For all non-negative real numbers a and b, and for every integer $q > 0$,

$$(ab)^{\frac{1}{q}} = a^{\frac{1}{q}} b^{\frac{1}{q}} .$$

G. Prove the following theorem.

> For every real number $a > 0$, and for all rational numbers x and y,
>
> $$(a^x)^y = a^{xy};$$
>
> for all non-negative rational numbers x and y,
>
> $$(0^x)^y = 0^{xy}.$$

[Hint: Use the theorems in Exercises 1 and 2 of Part F.]

H. Prove the following theorem.

> For all positive real numbers a and b, and for every rational number x,
>
> $$(ab)^x = a^x b^x;$$
>
> and likewise for all non-negative real numbers a and b and for every non-negative rational number x.

I. Apply the previous theorems to simplify each of the following. Give answers without using symbols for negative exponents.

I. (Cont.)

1. xy^3

2. a^2b^6

3. $4a^{\frac{2}{5}}c^4$

4. $8bc^2$

5. $27p^6$

6. $\left(x^3 + y^6\right)^{\frac{1}{6}}$

7. $x^{\frac{3}{2}} + x^{\frac{1}{2}}$

8. $2x^{\frac{3}{2}}$

9. $\dfrac{64t^9}{27a^{\frac{3}{2}}}$

10. $9a^3x^6$

11. $\dfrac{9}{a^2b^8}$

12. $\dfrac{x^6z^{12}}{26^{\frac{3}{2}}y^9}$

13. $\frac{1}{2}$

14. 2

Sample 1. $(x^2y^6)^{\frac{1}{3}}$ $[x \geq 0,\ y \geq 0]$

Solution. $(x^2y^6)^{\frac{1}{3}}$

$= = (x^2)^{\frac{1}{3}}(y^6)^{\frac{1}{3}}$ [Why?]

$= x^{\frac{2}{3}}y^2$. [Why?]

Sample 2. $(x^2)^{\frac{3}{10}} + 5x^{\frac{3}{5}}$ $[x \geq 0]$

Solution. $(x^2)^{\frac{3}{10}} + 5x^{\frac{3}{5}}$

$= x^{\frac{6}{10}} + 5x^{\frac{3}{5}}$

$= x^{\frac{3}{5}} + 5x^{\frac{3}{5}}$

$= 6x^{\frac{3}{5}}$

1. $\left(x^{\frac{1}{2}}y^{\frac{3}{2}}\right)^2$, $[x \geq 0,\ y \geq 0]$

2. $(a^3b^9)^{\frac{2}{3}}$, $[a \geq 0,\ b \geq 0]$

3. $\left(8a^{\frac{3}{5}}c^6\right)^{\frac{2}{3}}$

4. $\left(4b^{\frac{1}{4}}c^{\frac{1}{3}}\right)\left(2b^{\frac{3}{4}}c^{\frac{5}{3}}\right)$

5. $(81p^8)^{\frac{3}{4}}$

6. $(x^3 + y^6)^{\frac{1}{6}}$

7. $\left(x^{\frac{1}{2}}\right)^3 + x^{\frac{1}{2}}$

8. $\left(x^{\frac{1}{2}}\right)^3 + \left(x^3\right)^{\frac{1}{2}}$

9. $\left[\dfrac{16a^2t^6}{9a^3}\right]^{\frac{3}{2}}$

10. $(4a^2x^4)^{\frac{3}{2}} + (a^6x^{12})^{\frac{1}{2}}$

11. $(27a^{-3}b^{-12})^{\frac{2}{3}}$

12. $(26x^{-4}y^6z^{-8})^{-\frac{3}{2}}$

13. $4^{-\frac{1}{2}}$

14. $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$

(continued on next page)

J.	1.	False	2.	True
	3.	False	4.	False
	5.	True	6.	False
	7.	True	8.	False
	9.	False	10.	False
	11.	True	12.	True

I. (Cont.)

15. $\frac{1}{4}$

16. $\frac{1}{16}$

17. $\frac{1}{8^{\frac{1}{2}}}$ or $\frac{1}{2^{\frac{3}{2}}}$

18. $\frac{1}{3}$

19. $\frac{1}{27}$

20. $\frac{1}{5x^2}$

21. $\frac{1}{25x^4}$

22. 1

23. x

24. $\frac{1}{3x}$

25. $\frac{1}{a^8}$

26. $\frac{b^{\frac{1}{2}}}{a^{\frac{1}{3}}}$

27. $\frac{b}{a^{\frac{2}{3}}c^{\frac{4}{3}}}$

28. $\left[\frac{a^2b^2}{a^2+b^2}\right]^2$

29. $\frac{1}{a^{\frac{7}{3}}}$

30. $\frac{1}{8a^9}$

31. $x^{\frac{1}{a}}$

32. $x^{5n-1}y^{n+1}$

(continued on T. C. 54B)

15. $8^{-2/3}$

16. $8^{-4/3}$

17. $64^{-\frac{1}{4}}$

18. $81^{-\frac{1}{4}}$

19. $243^{-\frac{3}{5}}$

20. $(125x^6)^{-\frac{1}{3}}$

21. $\left(125x^6\right)^{-\frac{2}{3}}$

22. $\left[\left(132x^4a^3\right)^{\frac{5}{7}}\right]^0$

23. $(x^{-2})^{-\frac{1}{2}}$

24. $(27x^3)^{-\frac{1}{3}}$

25. $\left[\frac{1}{a^{-2}}\right]^{-4}$

26. $\left(\frac{b^{-3}}{a^{-2}}\right)^{-\frac{1}{6}}$

27. $\left[\frac{a^3b^{-3}}{ac^{-4}}\right]^{-\frac{1}{3}}$

28. $[a^{-2} + b^{-2}]^{-2}$

29. $\left(a^{\frac{1}{2}}a^{\frac{1}{3}}a^{-\frac{1}{4}}\right)^{-4}$

30. $(32a^{15})^{-\frac{3}{5}}$

31. $\left[[x^{-a}]^{\frac{1}{b}}\right]^{-b/a^2}$

32. $\frac{x^{2n+1}y^{-n+4}}{x^{-3n+2}y^{-2n+3}}$

J. Tell whether each of the following statements is true or false.

1. $5^0 = 0$

2. $6^{-5}6^5 = 1$

3. $7^{-3}7^{-5} = 7^{15}$

4. $2^{-1} = -2$

5. $-5^2 = -25$

6. $(3^5)^2 = 3^{5^2}$

7. $\frac{9^3}{9^5} = 9^{-5}9^3$

8. $\frac{1}{3^4 + 2^5} = 3^{-4} + 2^{-5}$

9. $(\pi + 7)^{-3} = \pi^{-3} + 7^{-3}$

10. $4^{-2} = -8$

11. $\left(\frac{6\pi^{-3}4^{-2}}{5^{-7}}\right)^0 = 5^0$

12. $\frac{\pi^{-1}}{7^{-1}} = \frac{7}{\pi}$

(continued on next page)

J. (Cont.)

13.	False	14.	False
15.	True	16.	True
17.	False	18.	False
19.	True	20.	True
21.	True	22.	True
23.	False	24.	False
25.	False	26.	True
27.	True	28.	False
29.	True	30.	True
31.	False	32.	True
33.	True	34.	False
35.	False	36.	True
37.	False	38.	False, $0^{-\frac{1}{7}}$ not defined

13. $\pi^{-3}\pi^{5} = \pi^{-15}$

14. $2^{-1} = -\frac{1}{2}$

15. $(-3)^{-2} = \frac{1}{9}$

16. $-3^{-2} = -\frac{1}{9}$

17. $5^{0} \times 5^{3} = 5^{0}$

18. $2^{3} \times 2^{5} = 4^{8}$

19. $0^{1} + 0^{2} = 0^{3}$

20. $1^{5} \times 1^{2} = 1^{10}$

21. $\pi^{7-3} = \frac{1}{\pi^{3-7}}$

22. $3^{1} + 3^{-1} = 10(3^{-1})$

23. $0^{5} = 1^{5}$

24. $0^{0} = 0^{1}$

25. $\left(\frac{1}{3}\right)^{-3} = \frac{1}{27}$

26. $\left(\frac{\pi}{\sqrt{2}}\right)^{-5} = \left(\frac{\sqrt{2}}{\pi}\right)^{5}$

27. $2^{-\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{1}{2}}$

28. $\pi^{2} + \pi^{3} = \pi^{5}$

29. $\left[\left(\frac{1}{3}\right)^{-\frac{1}{7}}\right]^{0} = 1$

30. $\frac{\pi^{-3}}{(\sqrt{5})^{-4}} = \frac{5^{2}}{\pi^{3}}$

31. $\sqrt[3]{4^{4}} = 4^{\frac{3}{4}}$

32. $7^{-\frac{1}{2}} \times 7^{\frac{1}{2}} = 1$

33. $0^{\frac{1}{2}} = 0$

34. $(\pi + \sqrt{2})^{-3} \div (\pi + \sqrt{2})^{3} = 1$

35. $10^{0.3} = \left(\frac{1}{10}\right)^{3}$

36. $\pi^{-\frac{1}{3}}(\pi^{\frac{4}{3}} + \pi^{0}) = \pi + \frac{1}{\sqrt[3]{\pi}}$

37. $10^{0.3} = 5^{0.6}$

38. $0^{-\frac{1}{7}} = 0$

The case in which $a > 0$ and $b < 0$ can be handled in a similar fashion. Suppose then that $a < 0$ and $b < 0$. In this case

$$\begin{aligned}(ab)^{\frac{p}{q}} &= ((-a)(-b))^{\frac{p}{q}} \\ &= (-a)^{\frac{p}{q}} \cdot (-b)^{\frac{p}{q}} \\ &= (-1)^{2p}(-a)^{\frac{p}{q}}(-b)^{\frac{p}{q}} && \text{[2p is even]} \\ &= a^{\frac{p}{q}} \cdot b^{\frac{p}{q}}. && \text{[Lemma 2]}\end{aligned}$$

This completes the proof that the three "laws of exponents" continue to hold in the cases covered by the defining principle at the bottom of page 2-56.

On the other hand, if p and s are both odd then

$$\left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = \left((-1)^p(-a)^{\frac{p}{q}}\right)^{\frac{r}{s}} \qquad \text{[Lemma 2 (q is odd)]}$$

$$= \left(-(-a)^{\frac{p}{q}}\right)^{\frac{r}{s}} \qquad \text{[p is odd]}$$

$$= (-1)^r\left((-a)^{\frac{p}{q}}\right)^{\frac{r}{s}} \qquad \text{[Lemma 2 } (-(-a)^{\frac{p}{q}} < 0, \text{ s is odd)]}$$

$$= (-1)^r(-a)^{\frac{pr}{qs}}$$

$$= (-1)^{pr}(-a)^{\frac{pr}{qs}} \qquad \text{[p is odd]}$$

$$= a^{\frac{pr}{qs}}. \qquad \text{[Lemma 2 (qs is odd)]}$$

III. Distributive law

For every a and b such that $a < 0$ or $b < 0$ and all integers p and q such that $q > 0$ and q is odd,

$$(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} \cdot b^{\frac{p}{q}}.$$

Proof: Suppose, first, that $a < 0$ and $b > 0$. Then

$$(ab)^{\frac{p}{q}} = (-1)^p(-ab)^{\frac{p}{q}}$$

$$= (-1)^p(-a)^{\frac{p}{q}} \cdot b^{\frac{p}{q}}$$

$$= a^{\frac{p}{q}} \cdot b^{\frac{p}{q}}.$$

(continued on T. C. 56F)

But $$(-1)^{p+r} \cdot \frac{1}{(-1)^{ps+qr}} = (-1)^{p(1-s)+p(1-q)} = 1,$$

since s and q are both odd. Hence

$$a^{\frac{p}{q}} \cdot a^{\frac{r}{s}} = a^{\frac{ps+qr}{qs}} .$$

II. Multiplication rule

For every $a < 0$ and all integers p, q, r, and s such that $q > 0$, $s > 0$, q is odd, and s is odd if p is odd,

$$\left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = a^{\frac{pr}{qs}} .$$

[The only case of the rule for multiplying exponents which is in question is that in which $a < 0$, and q is odd. If p is odd then $a^{\frac{p}{q}} < 0$, so we need only consider the case in which s is odd.]

Proof: Suppose, first, that p is even. Then

$$\begin{aligned}
\left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} &= \left((-1)^p(-a)^{\frac{p}{q}}\right)^{\frac{r}{s}} && \text{[Lemma 2 (q is odd)]} \\
&= \left((-a)^{\frac{p}{q}}\right)^{\frac{r}{s}} && \text{[p is even]} \\
&= (-a)^{\frac{pr}{qs}} && \text{[Rule for multiplying exponents in the case of a positive base]} \\
&= (-1)^{pr}(-a)^{\frac{pr}{qs}} && \text{[pr is even]} \\
&= a^{\frac{pr}{qs}} . && \text{[Lemma 2 (qs is odd)]}
\end{aligned}$$

(continued on T. C. 56E)

In order to prove that the rules for adding and multiplying exponents and the distributive law of exponentiation with respect to multiplication still hold it is convenient first to prove the following.

Lemma 2: For every $a < 0$, every integer p, and every odd integer $q > 0$,

$$a^{\frac{p}{q}} = (-1)^p(-a)^{\frac{p}{q}} .$$

[Note that it is only after question (ii) has been answered affirmatively that we are justified in writing

$$\text{`}a^{\frac{p}{q}}\text{'}.]$$

Proof: By Lemma 1, $a^{\frac{p}{q}} = \left(-1\right)^p\left(\sqrt[q]{-a}\right)^p$, and by the defining principle on page 2-47, $\left(\sqrt[q]{-a}\right)^p = \left(-a\right)^{\frac{p}{q}}$.

I. Addition rule

For every $a < 0$ and all integers p, q, r, and s such that $q > 0$, $s > 0$, and q and s are odd,

$$a^{\frac{p}{q}} \cdot a^{\frac{r}{s}} = a^{\frac{p}{q} + \frac{r}{s}} .$$

Proof: By Lemma 2, and the rule for adding exponents in the case of integral exponents, and in the case of a non-negative base,

$$a^{\frac{p}{q}} \cdot a^{\frac{r}{s}} = (-1)^{p+r}(-a)^{\frac{p}{q}}(-a)^{\frac{r}{s}} = (-1)^{p+r}(-a)^{\frac{ps+qr}{qs}} .$$

Again, by Lemma 2,

$$(-a)^{\frac{ps+qr}{qs}} = \frac{1}{(-1)^{ps+qr}}\, a^{\frac{ps+qr}{qs}} .$$

(continued on T. C. 56D)

in computing a^x. An affirmative answer to question (ii) will show that, for every integer x, the number a^x given by the defining principle is the same as that given by the inductive definition on page 2-15. [The question corresponding to (ii) in the case $a \geq 0$ was answered in the note at the top of page 2-48.]

An affirmative answer to (ii) is justified by the fact that, for every $a < 0$,

$$\sqrt[1]{a} = -\sqrt[1]{-a} = -(-a) = a\ .$$

In order to answer question (i) it is convenient first to prove the following.

Lemma 1: For every $a < 0$, every integer p, and every odd integer $q > 0$,

$$\left(\sqrt[q]{a}\right)^p = \left(-1\right)^p\left(\sqrt[q]{-a}\right)^p.$$

Proof: $\left(\sqrt[q]{a}\right)^p = \left(-\sqrt[q]{-a}\right)^p = \left(-1\right)^p\left(\sqrt[q]{-a}\right)^p$, by the distributive law of exponentiation (for integral exponents) with respect to multiplication.

Now, as to (ii), since $\frac{p}{q} = \frac{r}{s}$, $ps = qr$; and, since q and s are both odd, it follows that p and r are either both odd or both even, so $(-1)^p = (-1)^r$. Since $-a > 0$, it follows from the boxed theorem on page 2-39 that $\left(\sqrt[q]{-a}\right)^p = \left(\sqrt[s]{-a}\right)^r$. Hence,

$$\left(-1\right)^p\left(\sqrt[q]{-a}\right)^p = \left(-1\right)^r\left(\sqrt[s]{-a}\right)^r$$

and, by Lemma 1,

$$\left(\sqrt[q]{a}\right)^p = \left(\sqrt[s]{a}\right)^r.$$

(continued on T. C. 56C)

On T. C. 37A, Part H, it is shown that, for every real number $a < 0$ and every odd integer $q > 0$, there is a unique real number x such that $x^q = a$. In fact, it is there shown that the real number x such that $x^q = a$ is $-\sqrt[q]{-a}$:

For every real number $a < 0$ and every odd integer $q > 0$,

$$\sqrt[q]{a} = -\sqrt[q]{-a} .$$

[If $a < 0$ then $-a > 0$, and $\sqrt[q]{-a}$ is the principal qth root of the positive number $-a$, as introduced on page 2-36.]

For lack of time, we have not, in the text, established the consistency of the defining principle at the bottom of page 2-56, nor have we proved that the rules for adding and multiplying exponents and the distributive law for exponentiation with respect to multiplication, still hold. [The following discussion is for your background information and for any student who raises questions.]

As to the consistency of the defining principle, there are two questions to be asked.

(i) Is it the case that, for every $a < 0$ and all integers p, q, r and s such that $q > 0$, $s > 0$, q and s are odd, and $\frac{p}{q} = \frac{r}{s}$,

$$\left(\sqrt[q]{a}\right)^p = \left(\sqrt[s]{a}\right)^r \quad ?$$

[Compare the above with the boxed theorem on page 2-39.]

(ii) Is it the case that, for every $a < 0$ and every integer p,

$$\left(\sqrt[1]{a}\right)^p = a^p \quad ?$$

An affirmative answer to question (i) will affirm the internal consistency of the defining principle, i.e., it will assure us that a^x depends only on the numbers a and x and not on the particular integers p and q which are used

(continued on T. C. 56B)

RATIONAL NUMBER EXPONENTS AND NEGATIVE BASES

Consider powers with rational exponents. Up to now we have explained powers with negative bases only when the exponent was an integer. [We have also explained powers with positive bases and rational exponents, and we have shown that powers of 0 cannot be consistently defined except when they have non-negative exponents.] To complete our discussion of powers with rational exponents we must consider the case of powers with negative bases and non-integral rational exponents such as:

$$(-3)^{\frac{2}{3}} \qquad (-\pi)^{-\frac{1}{2}} \qquad (-5)^{-\frac{3}{5}}$$

You have seen [Part H on page 2-37] that, for every real number $a < 0$, and for every odd integer $q > 0$, there is a unique real number x such that $x < 0$ and $x^q = a$. That is, there is a unique number $\sqrt[q]{a}$ such that $\sqrt[q]{a} < 0$ and $\left(\sqrt[q]{a}\right)^q = a$. [For example, $\sqrt[3]{-8} = -2$ because $-2 < 0$ and $(-2)^3 = -8$.] You have also seen that for every real number $a < 0$, and for every even integer $q > 0$, there is no real number x such that $x^q = a$. [In a later unit we introduce a number system (complex numbers) for which we shall be able to prove the following theorem:

> For every complex number $a \neq 0$, and for every integer $n > 0$, the equation:
>
> $$x^n = a$$
>
> has exactly n complex roots.]

Just as we did on page 2-47, we give a general defining principle which covers the case of negative bases

> For every number $a < 0$, and for every rational number x for which there exists integers p and q such that $q > 0$, q is odd, and $x = \frac{p}{q}$,
>
> $$a^x = \left(\sqrt[q]{a}\right)^p.$$

The corollaries of the addition rule for exponents which are referred to are stated in Exercises 1 and 2 of Part D, on page 2-50. Their proofs, using the addition rule, I, on T. C. 56C, are the same as the proofs given on T. C. 50A.

* * *

1. If there were an integer p and an odd $q > 0$ such that $\frac{3}{18} = \frac{p}{q}$ then the odd number 3q and the even number 18p would be the same.

   But no number is both odd and even. Hence '$(-8)^{\frac{3}{18}}$' is not defined by the boxed principle.

2. Since $\frac{16}{24} = \frac{2}{3}$, 2 is an integer, and 3 is a positive odd integer, $(-9)^{\frac{16}{24}} = \left(\sqrt[3]{-9}\right)^2$. [In fact, $\left(\sqrt[3]{-9}\right)^2 = \left(-\sqrt[3]{9}\right)^2 = \left(\sqrt[3]{9}\right)^2 = 9^{\frac{2}{3}}$.]

3. Since $\frac{100}{50} = \frac{2}{1}$, 2 is an integer, and 1 is a positive odd integer, $\left(-12\right)^{\frac{100}{50}} = \left(\sqrt[1]{-12}\right)^2 = \left(-12\right)^2 = 144$. [This is an example of the situation covered in answering question (ii) on T. C. 56A.]

4. As in Exercise 1, if there were integers p and q such that $q > 0$ and q odd, and $\frac{p}{q} = \frac{21}{28}$, then there would be an integer which is both odd and even. Hence '$(-2)^{\frac{21}{28}}$' is not defined by the boxed principle.

Thus,

$$(-8)^{\frac{2}{3}} = \left(\sqrt[3]{-8}\right)^2 = (-2)^2 = 4 .$$

Although the defining principle does not entitle you to say that

$$(-8)^{\frac{4}{6}} = \left(\sqrt[6]{-8}\right)^4 ,$$

nevertheless, since $\frac{4}{6} = \frac{2}{3}$,

$$(-8)^{\frac{4}{6}} = (-8)^{\frac{2}{3}} = 4.$$

By virtue of the boxed principle, the rule for adding exponents (and its two corollaries), the rule for multiplying exponents, and the distributive principle for exponentiation over multiplication continues to hold.

Exercise. Which of the following exponentials can be defined by using the boxed principle.

1. $(-8)^{\frac{3}{18}}$ 2. $(-9)^{\frac{16}{24}}$ 3. $(-12)^{\frac{100}{50}}$ 4. $(-2)^{\frac{21}{28}}$

[Hint for Exercise 1: Since there do not exist integers p and q such that q is odd and > 0 and $\frac{3}{18} = \frac{p}{q}$ (Why?), the principle does not apply.]

2.07 Irrational real numbers as exponents.--Let us draw the locus of the equation:

$$y = 2^x .$$

The following table is useful in drawing the locus. Be sure you understand the entries.

x	2^x
-4	$2^{-4} = \frac{1}{2^4} = \frac{1}{16} = 0.0625$
-3.5	$2^{-3.5} = 2^{-\frac{7}{2}} = (\sqrt{2})^{-7} = \frac{1}{(\sqrt{2})^7} = \frac{1}{8\sqrt{2}} \overset{a}{=} 0.1$
-3	$2^{-3} = 0.125$
-2.5	$2^{-2.5} \overset{a}{=} 0.2$
-2	0.25
-1.5	0.4 (approximately)
-1	0.5
-0.5	0.7
0	1
0.5	1.4
1	2
1.5	$2^{1.5} = 2^{\frac{3}{2}} = (\sqrt{2})^3 = 2\sqrt{2} \overset{a}{=} 2.8$
2	4
2.5	5.7
3	8
3.5	11.3
4	16

through the points which are plotted in the figure on page 2-59, a curve which has "humps" then it is easy to see that there is a rational number x such that $(x, 2^x)$ is not a point of this curve. If one modifies the curve so that is passes through this point, but still has humps, then another point $(x, 2^x)$, with x rational, can be found which should be on the new curve, but isn't. Ultimately, the fact that every interval, however short, of the number line contains rational numbers, rules out the possibility that there be two smooth curves which contain all points $(x, 2^x)$, x rational. The property of the rational numbers just referred to is equivalent to the fact that there are rational numbers as close as one pleases to any given irrational number. For example, 3.14 differs from π by less than 0.002, 3.14159 differs from π by less than 0.000003, etc.

[2-59]

Correction: The points (2.5, 5.7), (3, 8), and (3.5, 11.3) are incorrectly plotted. Have students make corrections on figure.

* * *

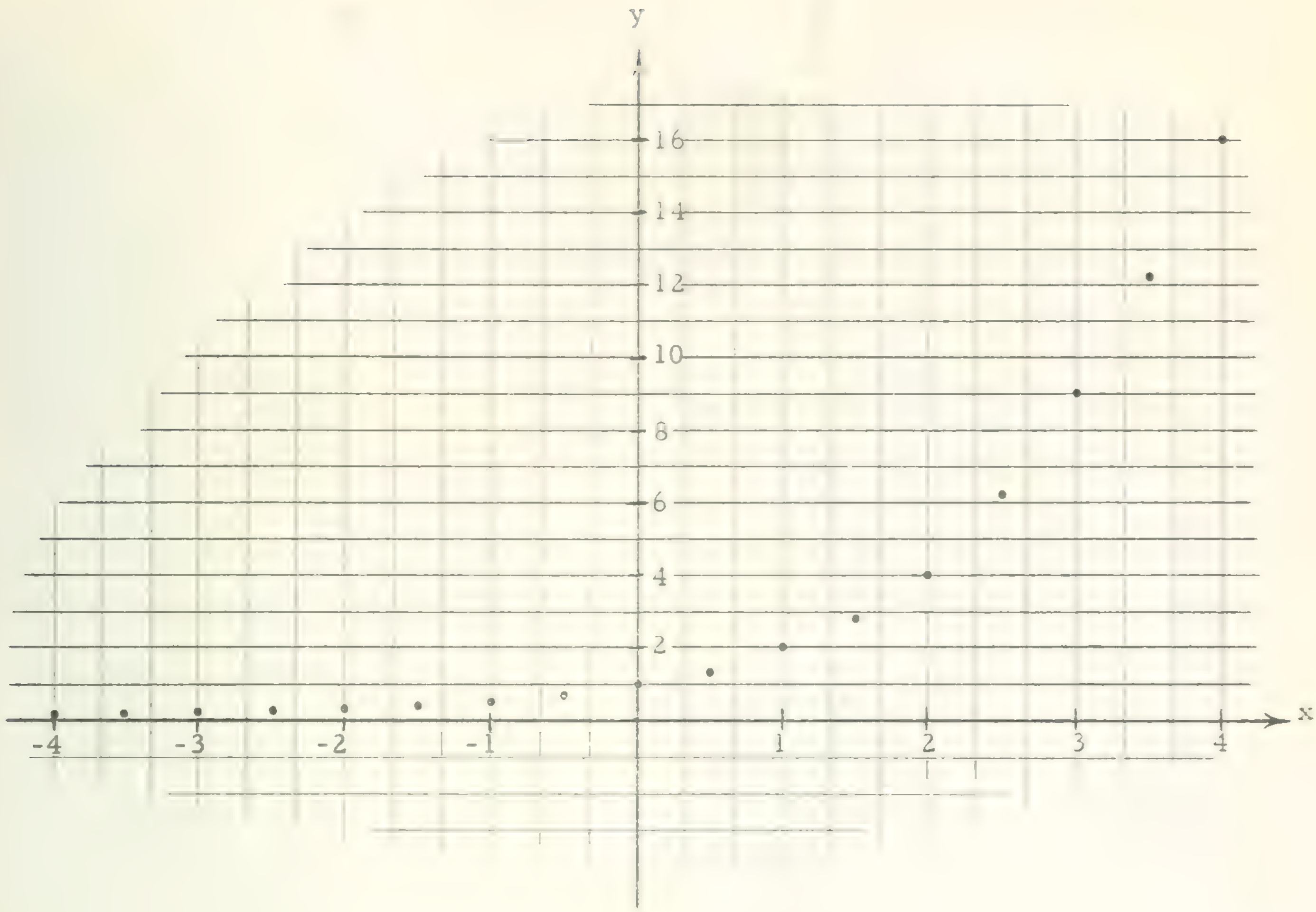
A. [In so far as powers of 2 have been defined, the locus of '$y = 2^x$' is not a curve in any accepted sense of the word 'curve', because there is no point on the locus which has an irrational abscissa. However, the locus is a subset of just one smooth curve. We shall use this fact to define powers of 2 with irrational exponents. Students should be aware that, previous to such a definition, the locus of '$y = 2^x$' consists of just the points $(x, 2^x)$ for rational x.]

1. Students have proved, in Exercise 1, Part D, on page 2-50, that, for every $a > 0$, and every rational x, $a^x \neq 0$. So the answer to the present exercise should be: No.

2. Here, again, the answer should be: No. Students may justify this by referring to the defining principle on page 2-47, the fact that principal roots of positive numbers are positive (see page 2-36), and the fact that powers with positive bases and integral exponents are positive. (The last follows, in the case of positive integral exponents, from the boxed theorem on page 2-32. This together with the first part of Exercise 1, Part D, page 2-19, settles the question for negative integral exponents. And, if $a \geq 0$, $a^0 = 1 > 0$.) This is a good exercise in tying together known results to obtain a needed conclusion (i.e., in proving a theorem).

3. A student's first answer may well be: Yes. However, he should be led to see that it is at least unlikely that there should be two such curves. (In fact, there is just one.) If one draws,

(continued on T. C. 59B)



EXERCISES

A. Note that these points appear to lie on a smooth curve. Sketch a smooth curve which passes through these points.

1. Do you think there exists a value of 'x' for which the corresponding value of 'y' is 0?

2. Do you think there exists a value of 'x' for which the corresponding value of 'y' is negative?

3. More than one smooth curve can be drawn through the given points. Do you think that two smooth curves could be drawn through all the points which belong to the locus? [Remember that only points with rational abscissas are under consideration. There is no point such as $(\pi, 2^{\pi})$ in this locus.]

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B. The easiest 10 points to plot are $(-\frac{2}{3}, 0.63)$, $(-\frac{1}{3}, 0.79)$, $(\frac{1}{3}, 1.26)$, $(\frac{2}{3}, 1.58)$, $(\frac{4}{3}, 2.52)$, $(\frac{5}{3}, 3.16)$, $(\frac{7}{3}, 5.04)$, $(\frac{8}{3}, 6.31)$, $(\frac{10}{3}, 10.08)$, and $(\frac{11}{3}, 12.62)$.

C. Each of the loci in Exercises 1 and 3 is symmetric to the other with respect to the y-axis. The locus in Exercise 5 is the same as that in Exercise 2. The locus in Exercise 6 (like each of the other loci) contains the point (0, 1); each of its remaining points is on the x-axis.

* * *

Correction: In each of the figures at the bottom of the page, the scales are intended to be the same on both axes. The locus of '$y = 2^x$' is correctly drawn, but that of '$y = 0.5^x$' is not. The latter should be symmetrical to the former with respect to the y-axis. Ask your students to look at the two loci and see whether they "look right". Let them bring out the fact that if the same scale is intended for both x-axes, then it can't be the case that both graphs are correct.

B. Use the fact that $\sqrt[3]{2} \doteq 1.26$ and plot at least 10 more points which belong to the locus of '$y = 2^x$'. Use the diagram in the text.

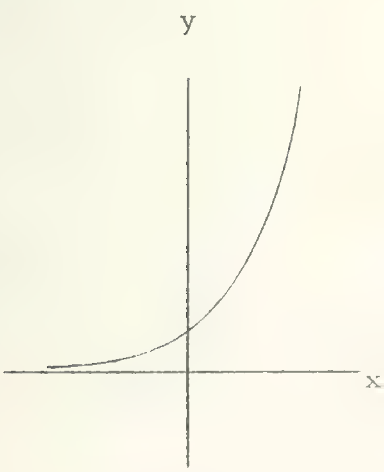
C. Draw the locus of each of the following equations.

1. $y = 3^x$; values of 'x' between -3 and 3.
2. $y = 0.5^x$; values of 'x' between -4 and 4.
3. $y = 3^{-x}$; values of 'x' between -3 and 3.
4. $y = 1^x$; values of 'x' between -5 and 5.
5. $y = 2^{-x}$; values of 'x' between -4 and 4.
6. $y = 0^x$; values of 'x' from 0 to 4.

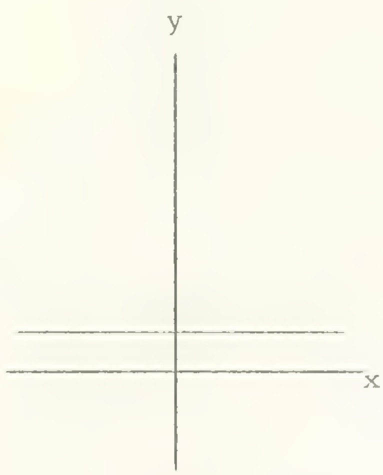
You may have guessed from the preceding exercises that if, for some number $a > 0$, you plot points belonging to the locus of the equation:

$$y = a^x \qquad (x \text{ \underline{rational}}),$$

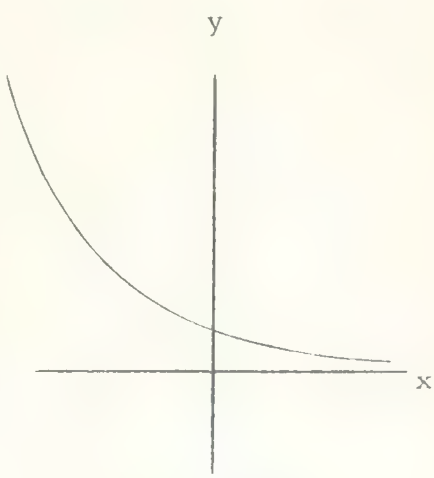
then these points will appear to lie on a smooth curve which is like one of the following:



$y = 2^x$



$y = 1^x$



$y = 0.5^x$

Answer to question: Up to now only powers with rational exponents have been defined.

* * *

In the discussion on this page we are hampered by the fact that the students who are at present studying this unit have not had experience with the concept of a function as a set of ordered pairs, no two of which have the same first component. If we could use this concept then we would substitute, in the text, 'smooth function' for 'smooth curve', and 'exponential function for the base a' for 'exponential curve for the base a'. Then characteristic (i) would say that each exponential function is a function (no two ordered pairs have the same first component), and is defined for every real number.

As it is, 'smooth curve' is made to carry more than its share of meaning. For example, a circle is, intuitively, about as smooth as a curve can be, yet it does not have characteristic (i). If students bring this up (and it might be well if they did), explain to them that for some purposes one might want to define 'smooth curve' in another way, omitting characteristic (i), (and perhaps adding other characteristics), but that we are interested here in curves which do have characteristics (i) and (ii), and couldn't think of a better phrase to apply to them than 'smooth curve'. We're sorry.

[Question: Why can you use only rational values of 'x'?]

It can be proved that, for every real $a > 0$,

(1) there is at least one smooth curve which for every rational number x, contains the point (x, a^x), and

(2) there is at most one smooth curve which, for every rational number x, contains the point (x, a^x).

You will find proofs of (1) and (2) in more advanced mathematics courses. For the present we shall accept them. For each real $a > 0$, we shall speak of the curve whose existence is insured by (1) and whose uniqueness is insured by (2) as the exponential curve for the base a.

The proofs of (1) and (2) depend on giving a precise meaning to the phrase 'smooth curve'. Although we shall not give a precise definition we can give a sufficiently clear idea of the concept by stating two of the characteristics of a smooth curve, as illustrated by the exponential curve for the base 2.

(i) For every real number x, there is a unique y such that the point (x, y) is on the exponential curve. [This means, in geometric terms, that every line perpendicular to the x-axis intersects the exponential curve in one and only one point.]

(ii) A second characteristic of a smooth curve can, like the first, be described by saying that a certain property holds for every real number. Rather than expressing the property itself we shall tell you what we mean when we say that the property holds for the real number $\sqrt{3}$:

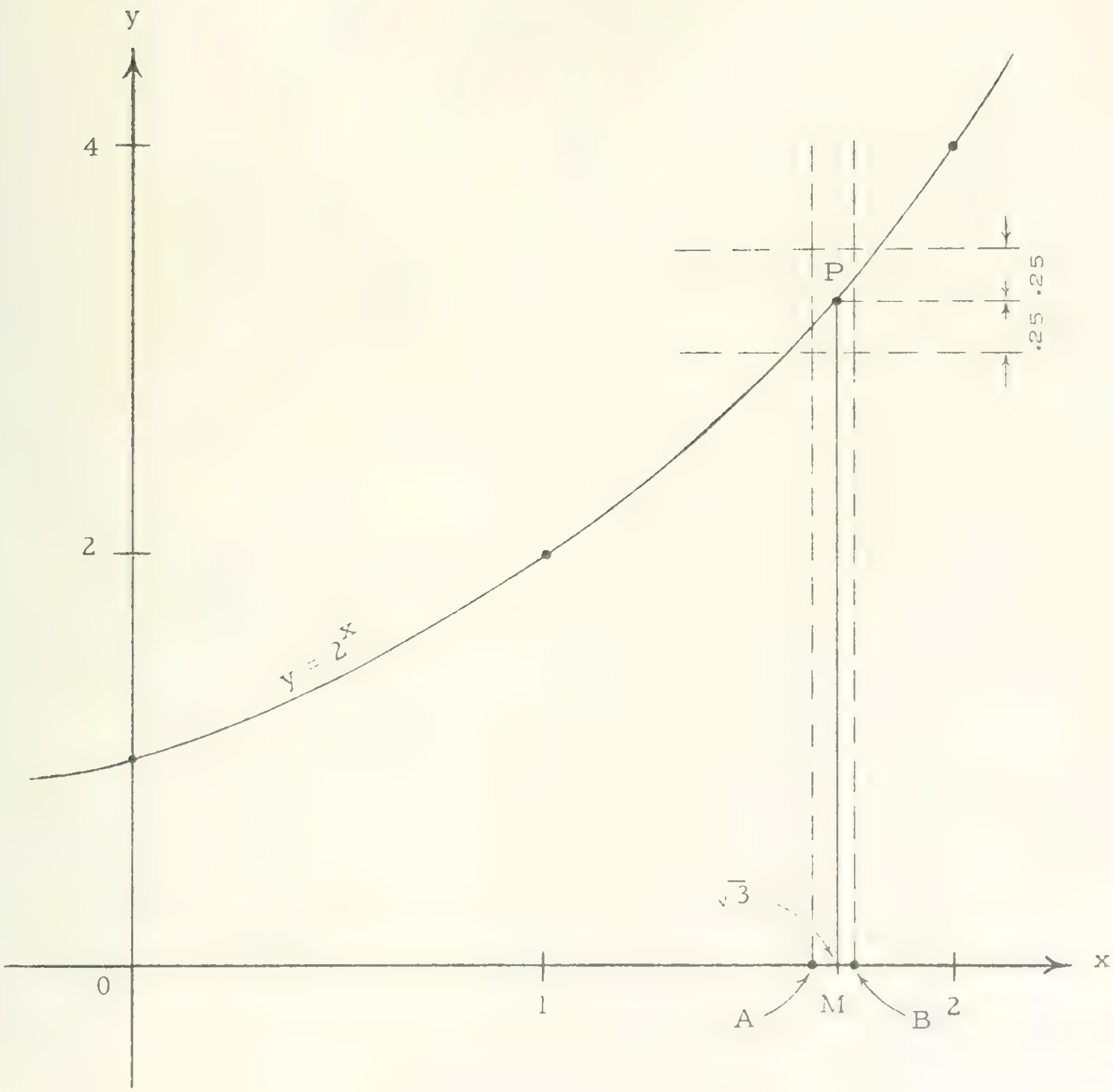
> There is a number y such that $(\sqrt{3}, y)$ is a point in the exponential curve for the base 2. This number y is approximated as closely as you wish by 2^x if x is any rational number sufficiently close to $\sqrt{3}$.

The points A and B might have been chosen farther from M. In fact, for A we might have taken the projection A′ (not shown on figure), of the intersection of the exponential curve with the lowest of the three horizontal dotted lines, and for B the projection B′, of the intersection of the curve with the highest of these lines. The point is that we want to show that there <u>are</u> points A and B such that M belongs to the interval $\overline{AB}$ and such that, for <u>every</u> rational number x for which (x, 0) belongs to $\overline{AB}$,

$$\left|2^{x} - m(\overset{\circ\!\!-\!\!-\!\!\circ}{MP})\right| < 0.25.$$

<u>Any</u> point A in $\overset{\circ\!\!-\!\!-}{A'M}$ and any point B in $\overset{-\!\!-\!\!\circ}{MB'}$ will do. We hope that students will see that this is the case. We feared that using A′ and B′ might tend to hide this from them.

In geometric terms, this means the following:



In view of (i) there is a point P in the locus of '$y = 2^x$' whose abscissa is $\sqrt{3}$. Our job is to approximate the ordinate of P [that is, $m(\overset{\circ\!-\!\circ}{MP})$]. Suppose we want to approximate this number with an error less than 0.25. We can draw a horizontal band which is "centered on" P and which is 0.50 wide. We wish to find a rational number x such that

$$|2^x - m(\overset{\circ\!-\!\circ}{MP})| < 0.25.$$

The boxed defining principle and characteristic (ii) make it possible to find decimal approximations to powers with irrational exponents. For example,

$$\sqrt{3} \overset{a}{=} 1.7 = \frac{17}{10}$$

so

$$2^{\sqrt{3}} \overset{a}{=} 2^{1.7} = \left(\sqrt[10]{2}\right)^{17}.$$

Now

$$\sqrt[10]{2} \overset{a}{=} 1.0718,$$

and

$$\left(\sqrt[10]{2}\right)^{17} \overset{a}{=} 3.24.$$

Hence,

$$2^{\sqrt{3}} \overset{a}{=} 3.24.$$

[A closer approximation to $\sqrt{3}$ is 1.732 and the corresponding approximation to $2^{\sqrt{3}}$ is, approximately, 3.323.]

* * *

We have seen that a power with a negative base and a <u>rational</u> exponent can be defined <u>only</u> if the exponent is the quotient of an integer by an odd integer. If, in this case, it is not even possible to use all rational numbers as exponents there is obviously very small likelihood that we could use irrational exponents with negative bases. [Even in the complex number system, where any rational number can be used fairly simply as an exponent for any real base, irrational exponents lead to complications.]

Characteristic (ii) of a smooth curve tells us that the above inequality will be satisfied by every rational number x for which

$$|x - \sqrt{3}| \text{ is sufficiently small.}$$

To say that $|x - \sqrt{3}|$ is <u>sufficiently</u> small means that points with abscissa x lie in a sufficiently narrow vertical band centered on the locus of '$x = \sqrt{3}$'. You can see such a vertical band in the diagram. It is "sufficiently narrow" because the locus of '$y = 2^x$' does not intersect the horizontal edges of the rectangle formed by the two bands. Clearly, any rational value of 'x' in $\overset{\circ\!-\!\circ}{AB}$ satisfies the inequality:

$$|2^x - m(\overset{\circ\!-\!\circ}{MP})| < 0.25.$$

If we want to ensure a smaller error in approximating $m(\overset{\circ\!-\!\circ}{MP})$ by values of '2^x' we need only pick a narrower horizontal band; characteristic (ii) assures us that we can find a vertical band which will give appropriate rational values of 'x'.

Characteristic (ii) of an exponential curve will be of use to us in a later section of this unit. At present we want to use characteristic (i). That every exponential curve has characteristic (i) suggests the following explanation of powers with <u>real</u> exponents (including <u>irrational</u> exponents).

For every real a > 0 and every real x, a^x is <u>the</u> real number y such that the point (x, y) is in the exponential curve for the base a.

Thus, for example, '$2^{\sqrt{3}}$' is a name for $m(\overset{\circ\!-\!\circ}{MP})$ [See above diagram]. Since the exponential curve for the base a contains the point (x, a^x) for every rational number x, the boxed explanation agrees with our previous explanation of powers with rational exponents [See page 2-47].

For obvious reasons we shall not consider powers with negative bases and irrational exponents [What are the reasons?]. However, it is convenient to complete the boxed statement by adding:

Students should be warned that not everyone defines '0^0' to be a name for the number 1. [In the usual treatments of differential calculus, the symbol '0^0' is used in an entirely different way, not as a numeral at all, but to refer to a type of problem which is treated under the subject heading: Indeterminate forms. This use of '0^0' in this sense, as well as the phrase 'indeterminate form', is inexcusable except on historical grounds.] However, those who do use '0^0' as a numeral use it as a name for 1, and many textbook writers (including all those who write about infinite series) use '0^0' as a name for 1; often, apparently, without realizing that they are doing so.

* * *

The proofs that the laws of exponents hold in the case of irrational exponents are usually delayed until students are at the graduate college level. At any rate, we won't try to give them here! (Maybe next year---.)

For every real number $x > 0$,

$$0^x = 0;$$

and

$$0^0 = 1.$$

By using characteristic (ii) of exponential curves one can show that the rule for adding exponents [‘‘$a^x a^y = a^{x+y}$’’] together with its corollaries, [‘‘$a^{-x} = \frac{1}{a^x}$’’, ‘‘$\frac{a^x}{a^y} = a^{x-y}$’’], the rule for multiplying exponents [‘‘$(a^x)^y = a^{xy}$’’], and the distributive principle for exponentiation over multiplication [‘‘$(ab)^x = a^x b^x$’’] hold in all cases covered by the defining principles above.

Your students will need to use the following theorem when they solve these equations.

> For every real number a, and every real number b,
>
> if $5^a = 5^b$ then $a = b$.

This theorem is proved on pages 2-88 and 2-99. You should assume this theorem here. Of course, if your students <u>want</u> to prove the theorem at this point it is all right for them to do so.

You will realize that interchanging abscissas and ordinates changes an exponential curve into a logarithm curve (if $y = 10^x$, then $x = \log_{10} y$). Students will use the exponential curves on pages 2-65 and 2-67 as substitutes for tables of logarithms to the bases 5 and 7, respectively. Of course, they should not, at this point, be told anything about logarithms. The whole point of our use of exponential curves is to furnish a painless and effective approach to logarithms. This approach would be spoiled if students were told (before page 2-85) that they are "really using logarithms". If some of your students have heard about logarithms from elder brothers or parents, you may have to do some judicious "sitting on them" lest they "corrupt" the others.

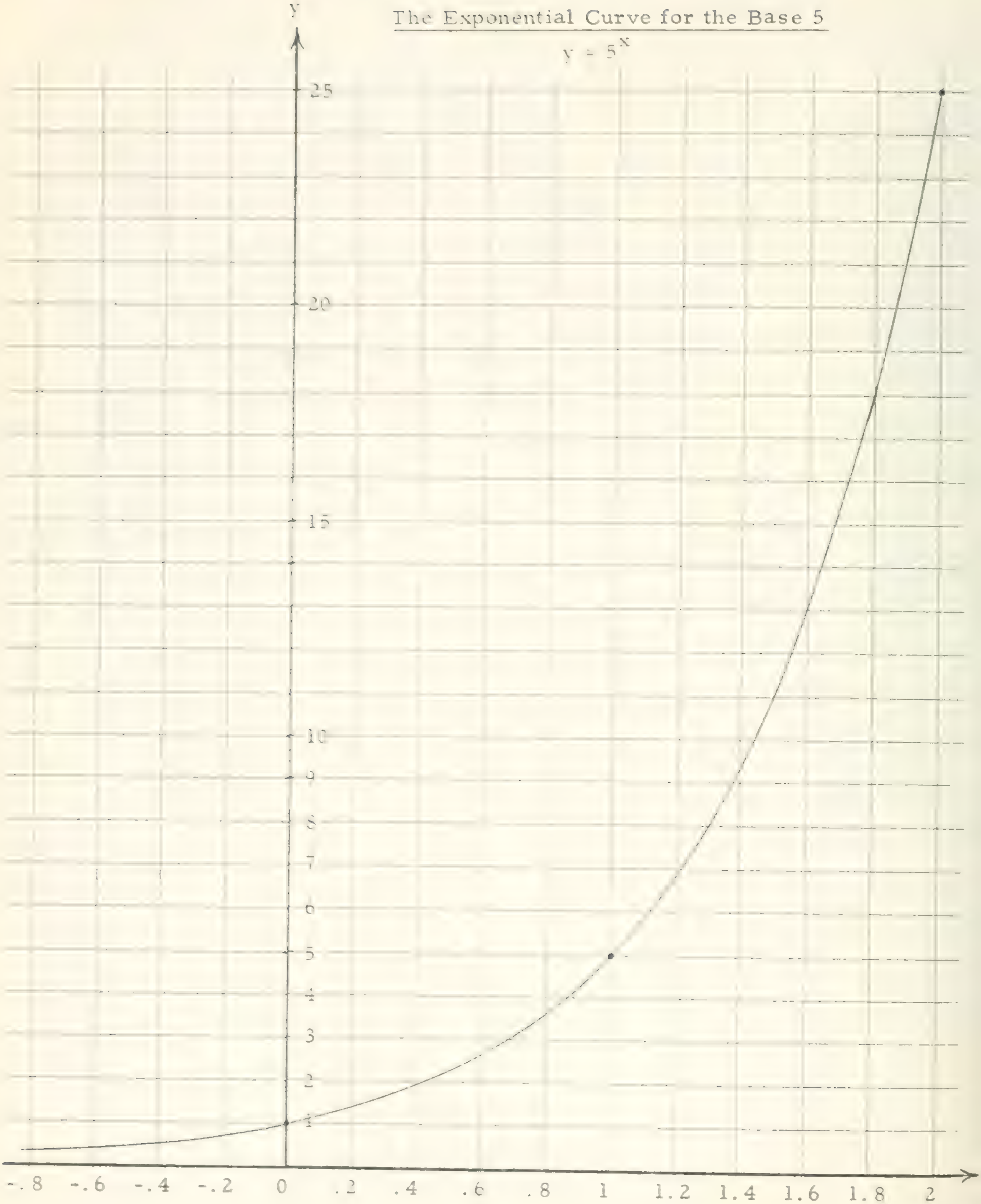
EXERCISES

A. 1. 2 | 2. 1.78
3. 9.5 | 4. 1.9
5. 0.84 | 6. 0.84
7. 0.62 [See Sample 3.] | 8. No root
9. 91 [$n = 25 \times 5^{0.8}$.] | 10. 625
11. 157 [$g \stackrel{a}{=} 125 \times 5^{0.14}$.] | 12. 9.8
13. 1.7
14. -2 [$2z = z - 2$.]

[Note that Exercises 5 and 6 have the same answer. Ask students why this is so. Answer: If $5^{2k-1} = 3$ then $5^{2k} = 3.5$, so $25^k = 15$, and conversely.]

The Exponential Curve for the Base 5

$y = 5^x$



EXERCISES

A. The drawing on page 2-65 represents a part of the exponential curve for the base 5. Use the curve to find an approximation (not in exponential or radical form) to the root of each of the following equations.

Sample 1. $5^{-0.2} = u$

Solution. Find the ordinate of that point on the curve whose abscissa is -0.2. The ordinate is approximately 0.75. Therefore, $5^{-0.2} \overset{a}{=} 0.75$.

Sample 2. $5^{v} = 10$

Solution. Find the abscissa of the point whose ordinate is 10. The abscissa is approximately 1.43. Therefore, $5^{1.43} \overset{a}{=} 10$.

Sample 3. $25^{r} = 75$

Solution.

$$
\begin{aligned}
(5^2)^r &= 75 \\
5^{2r} &= 75 \\
\frac{5^{2r}}{5^2} &= \frac{75}{5^2} \\
5^{2r-2} &= 3 \\
2r - 2 &\overset{a}{=} 0.68 \\
2r &\overset{a}{=} 2.68 \\
r &\overset{a}{=} 1.34
\end{aligned}
$$

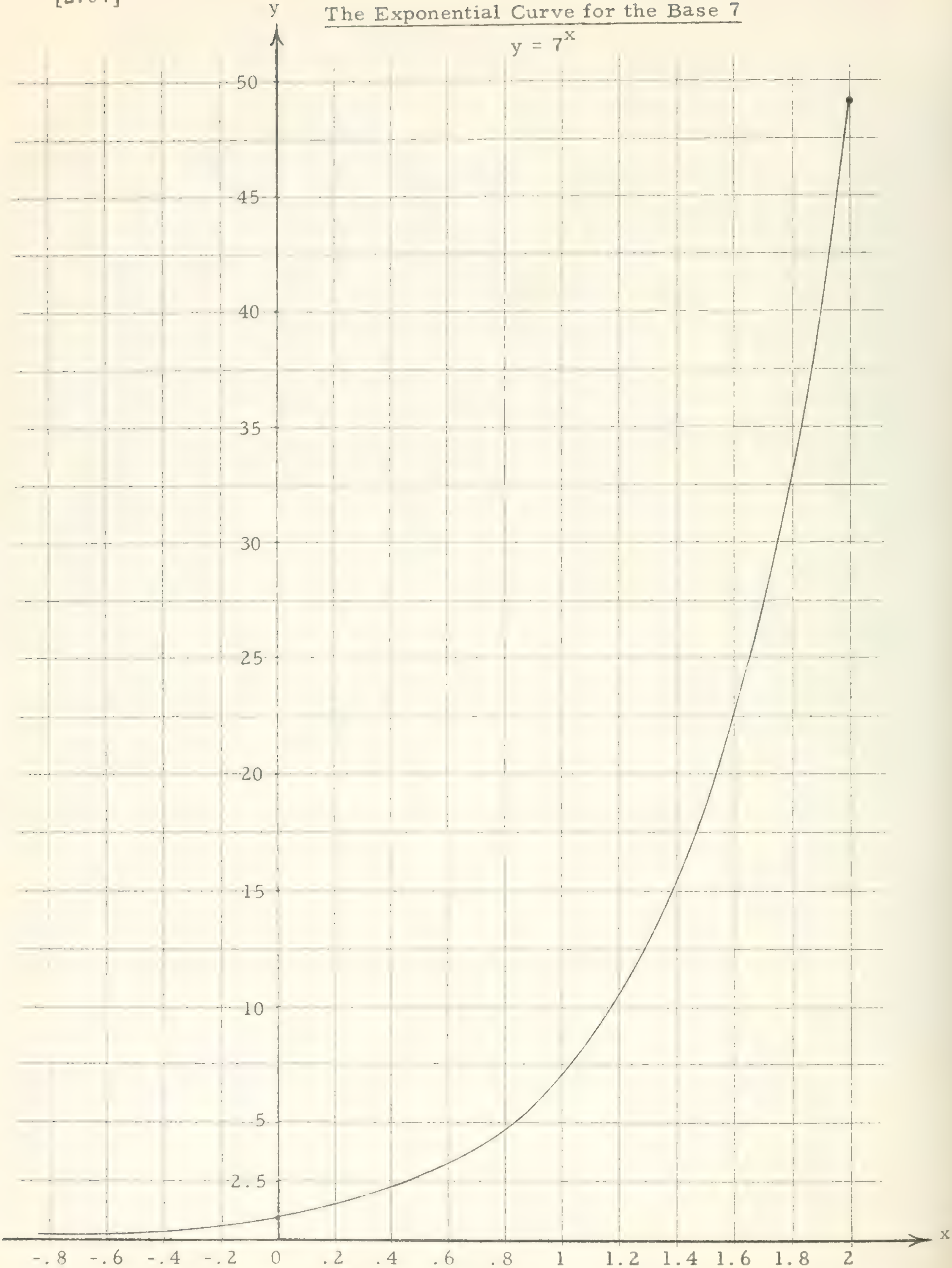
1. $5^{a} = 25$
2. $5^{b} = 17.5$
3. $5^{1.4} = c$
4. $5^{0.4} = d$
5. $25^{k} = 15$
6. $5^{2k-1} \quad 3$
7. $125^{m} = 20$
8. $5^{a} = -2$
9. $5^{2.8} = n$
10. $5^{4} \quad p$
11. $5^{\pi} = g$
12. $5^{\sqrt{2}} = h$
13. $625^{2x-3} = 18$
14. $25^{z} \quad 5^{z-2}$

B. 1.	1.8	2.	19
3.	1	4.	0.95
5.	2.4	6.	0
7.	-0.2	8.	$\frac{2}{3}$

[Students can obtain exact answers for Exercises 3, 6, 7, and 8 without using the curve. In using the picture of the exponential curve for the base 7 students will not obtain so accurate results as in Part A. This is due largely to the difference in scales on the y-axes.]

The Exponential Curve for the Base 7

$y = 7^x$



B. The drawing on page 2-67 represents a part of the exponential curve for the base 7. Proceed as in Part A.

1. $7^k = 30$
2. $7^{1.5} = m$
3. $7^2 = 49^x$
4. $49^t = 40$
5. $7^k = 98$
6. $7^{3m+2} = 49$
7. $7^{5s} = \frac{1}{7}$
8. $49^{2-3r} = 7^{3r-2}$

C. You can use the drawing on page 2-65 to reduce the labor of multiplying and dividing. Your answers will be approximate but quite frequently an approximate answer is all that is required in solving a problem. [You could improve the accuracy of your answers by using a drawing made to a larger scale.] The first three samples do not show a reduction in the labor of computing. They show you a use for an exponential curve:

Sample 1. 7×3

Solution.

$$\begin{aligned} 7 \times 3 &\stackrel{a}{=} 5^{1.21} \times 5^{0.68} \\ &= 5^{1.21 + 0.68} \\ &= 5^{1.89} \\ &\stackrel{a}{=} 21 \end{aligned}$$

Sample 2. $7.2 \div 3.5$

Solution.

$$\begin{aligned} \frac{7.2}{3.5} &\stackrel{a}{=} \frac{5^{1.23}}{5^{0.78}} \\ &= 5^{1.23 - 0.78} \\ &= 5^{0.45} \\ &\stackrel{a}{=} 2.1 \end{aligned}$$

C. 1. 25.2 2. 9.1

3. 3.2 4. 2.5

5. 3750 6. 2.64

7. 55.6 8. 7.35

[As to Exercise 5, Part C, $15.4 \stackrel{a}{=} 5^{1.7}$, so $(15.4)^3 \stackrel{a}{=} 5^{5.1}$. Now $5^{5.1} = 5^5 \cdot 5^{0.1}$, $5^4 \cdot 5^{1.1}$, $5^3 \cdot 5^{2.1}$, etc. Students may obtain somewhat different results according to which of these expressions they use.]

D. [As noted on T. C. 68A, answers obtained in Part D will, in general, be less accurate than those obtained in Part C.]

Sample 3. 9.7^3

Solution.
$$\begin{aligned} 9.7^3 &\overset{a}{=} (5^{1.41})^3 \\ &= 5^{4.23} \\ &= 5^4 \times 5^{0.23} \\ &= 625 \times 1.4 \\ &= 875 \end{aligned}$$

Sample 4. $\dfrac{98.6 \times 84.7}{3.89}$

Solution.
$$\begin{aligned} \frac{98.6 \times 84.7}{3.89} &= \frac{9.86 \times 8.47 \times 10^2}{3.89} \\ &\overset{a}{=} \frac{5^{1.42} \times 5^{1.33} \times 10^2}{5^{0.85}} \\ &= 5^{1.42+1.33-0.85} \times 10^2 \\ &= 5^{1.90} \times 10^2 \\ &\overset{a}{=} 21.4 \times 10^2 \\ &= 2140 \end{aligned}$$

Use the methods illustrated in the samples to simplify each of the following expressions.

1. 7.2×3.5
2. 18.6×0.49
3. $20.2 \div 6.4$
4. $58.8 \div 23.6$
5. $(15.4)^3$
6. $\sqrt[3]{18.6}$
7. $\dfrac{35.8 \times 143.7}{92.6}$
8. $\dfrac{1521 \times 87.2}{638 \times 28.3}$

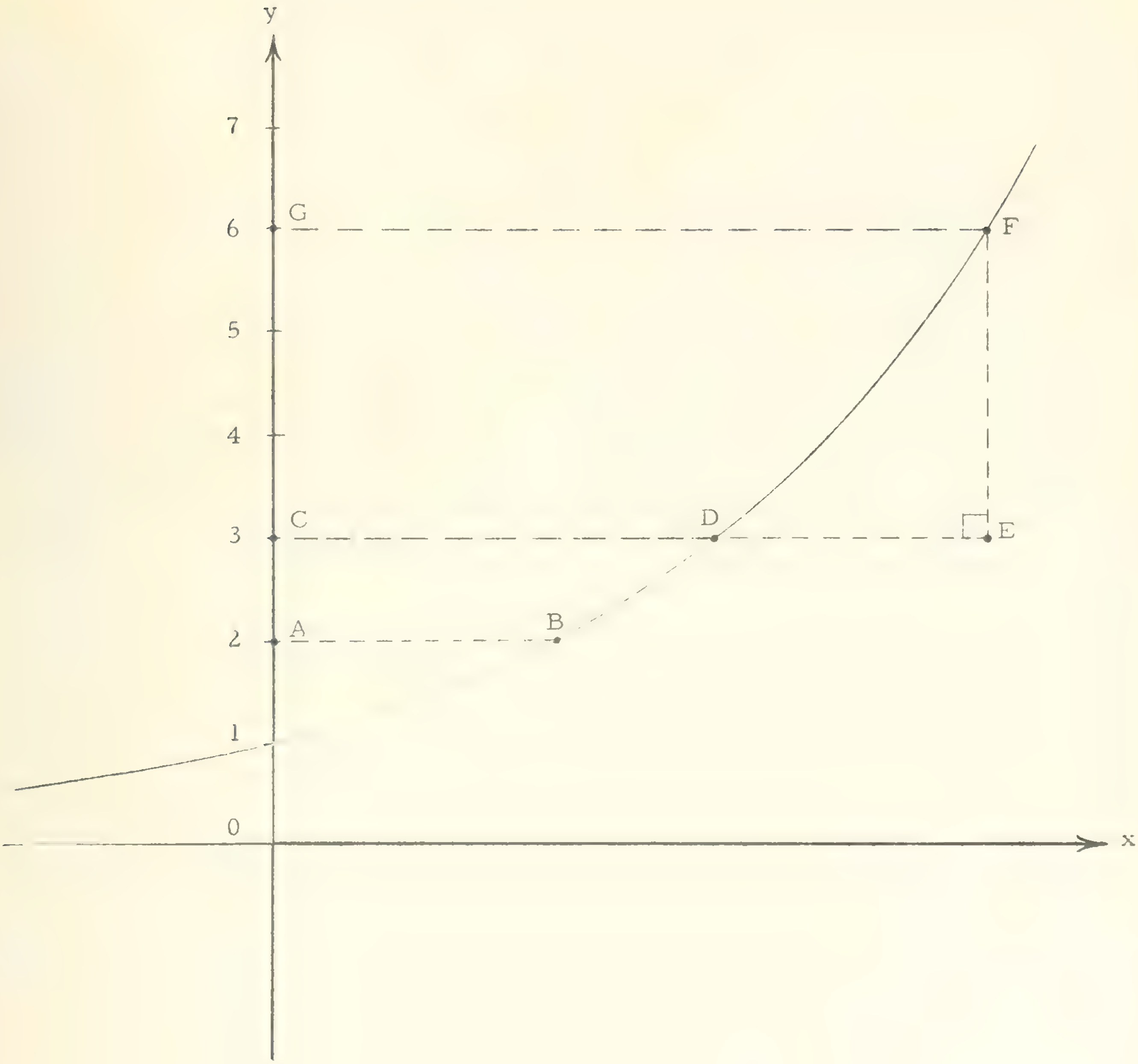
D. Use the drawing on page 2-67 and repeat Part C.

E. Parts C and D illustrated the fact that you can use an exponential curve for the base 5 or for the base 7 as an aid in carrying out computations. There is a quicker method for using an exponential curve to assist in computations. We illustrate this method for a simple case, that of finding the product of 2 by 3.

E. Note that the base is not given for the exponential curve shown in the figure. All that one needs to know in order to use the curve to assist in computations is that it _is_ an exponential curve.

Granting that the figure does represent an exponential curve, (actually, it is not a very accurate drawing) it is impossible to determine from the figure what its base is. This depends entirely on the scale which is used for the x-axis. For example, _if_ the projection of B on the x-axis is taken to represent the point (1, 0) then the figure pictures the exponential curve for the base 2. If the same point is taken as representing (2, 0) then the base is $\sqrt{2}$. [If $b > 0$, and $b^2 = 2$, then $b = \sqrt{2}$.] The arrow at the end of the x-axis indicates that the base for the curve is > 1. If, however, we disregard this and take the projection of D on the x-axis to represent the point (-1, 0) then the figure represents the exponential curve for the base $\frac{1}{3}$.

Try to bring out this information in class; for example, by asking students if they can guess what the base of the pictured exponential curve is, or asking what they would need to know in order to find this out. After it has become clear that it is the scale for the x-axis which is in question, students can estimate from measurements made on the picture what the base is if the scale for the x-axis is the same as that indicated for the y-axis. [In this case the base is 1.13, approximately.]



The diagram represents a piece of an exponential curve. Suppose b is the base for this exponential curve. Then

$$b^{m(\overset{\circ\!-\!\circ}{AB})} = 2 \quad \text{and} \quad b^{m(\overset{\circ\!-\!\circ}{CD})} = 3.$$

Therefore,

$$\begin{aligned} 2 \times 3 &= b^{m(\overset{\circ\!-\!\circ}{AB})} \times b^{m(\overset{\circ\!-\!\circ}{CD})} \\ &= b^{m(\overset{\circ\!-\!\circ}{AB}) + m(\overset{\circ\!-\!\circ}{CD})}. \end{aligned}$$

If Z is the point such that

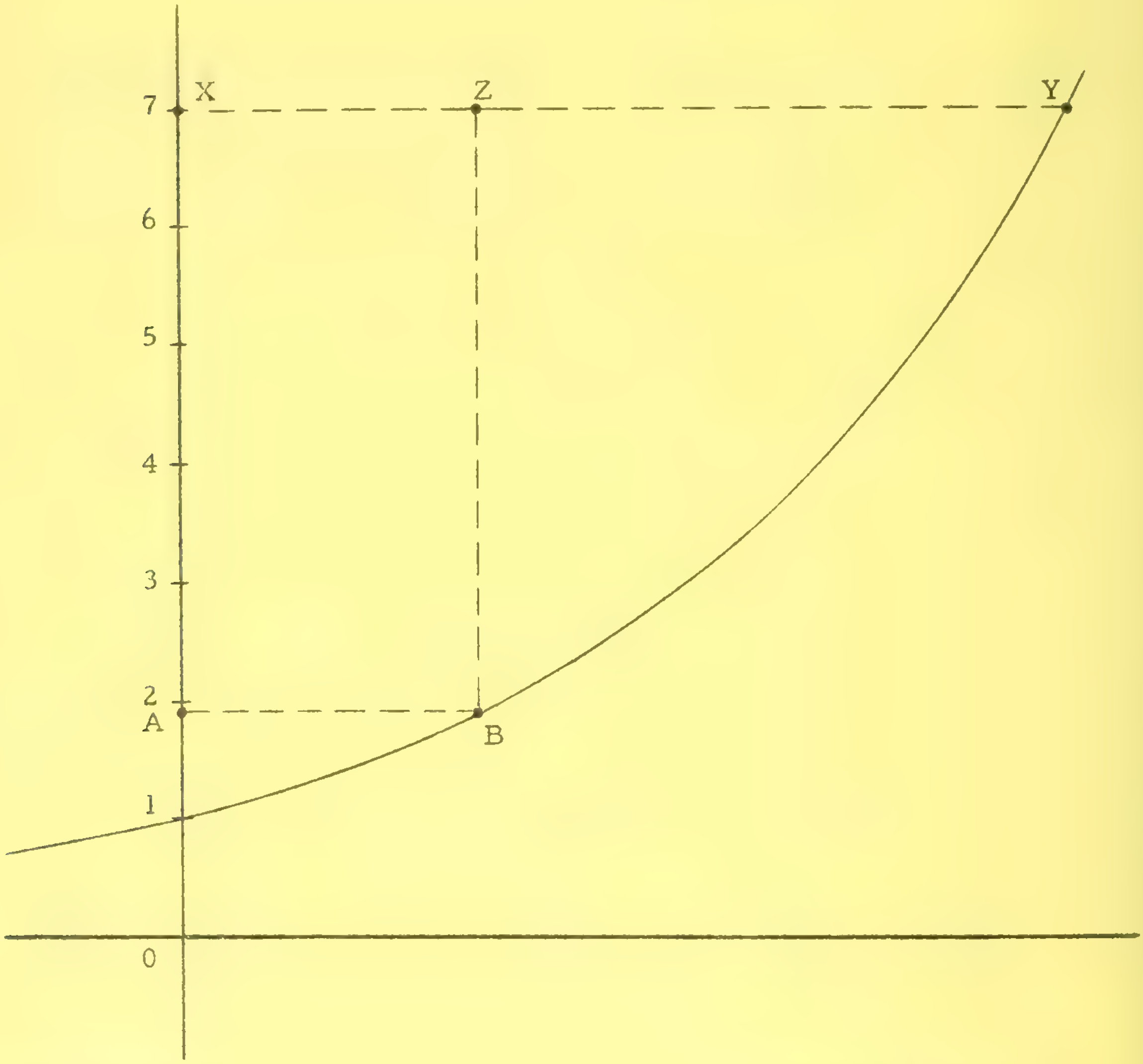
$$m(\overset{\circ\!\!-\!\!\circ}{XZ}) = \frac{1}{3} m(\overset{\circ\!\!-\!\!\circ}{XY}) \text{ then}$$

$$b^{m(\overset{\circ\!\!-\!\!\circ}{XZ})} = \sqrt[3]{7} \ .$$

Then by jumping from Z to the curve, and then over to the y-axis you can find the number corresponding to the cube root of 7. Here is a good chance to review the ruler and compass construction for dividing a segment into any number of congruent segments.

This geometric method may also be used to simplify any exponential which has a rational exponent.

Here is a sample which shows how the method is used.



Suppose you want an approximation for $\sqrt[3]{7}$. If b is the base of the exponential curve shown above then

$$b^{m(\overleftrightarrow{XY})} = 7.$$

(continued on T. C. 71C Supplement)

As pointed out on T. C. 70A, one does not have to know the base for the exponential curve in order to use the geometric method of computing. Of course one could not use the exponential curve for the base 1 (except for multiplying and dividing 1 by 1), but whatever the base of the curve pictured on page 2-70 may be it is not 1.

1. 20	2. 21	3. 15
4. 18	5. 24	6. 48
7. 3	8. 4	9. 0.5
10. 3	11. 4	12. 0 [But Exercise 12 cannot be solved by using an exponential curve!]

If $m(\overline{DE}) = m(\overline{AB})$ then $m(\overline{CE})$ is the abscissa of the point F on the curve whose ordinate is $b^{m(\overline{AB}) + m(\overline{CD})}$. You see that the ordinate of F is 6. Therefore,

$$2 \times 3 = 6.$$

By "adding" the segment $\overline{AB}$ to the segment $\overline{CD}$, jumping to the curve, and then over to the y-axis, you can find the product of the numbers corresponding to the points A and C.

Do you have to know the base for the exponential curve in order to use this geometric method of computing? [Could you use the exponential curve for the base 1?]

Use this geometric method with one or the other of the two exponential curves (pages 2-65 and 2-67) to simplify the following expressions.

1. 4×5	2. 3×7	3. 5×3
4. 2×9	5. 8×3	6. 6×8
7. $6 \div 2$	8. $20 \div 5$	9. $7 \div 14$
10. 6×0.5	11. 20×0.2	12. 3×0

F. You have seen how exponential curves may be used in carrying out computations. Of course, when you use exponential curves you must be satisfied with approximate answers. You can improve the accuracy of your answers by using an exponential curve which has been drawn to a larger scale. For example, let us use the exponential curve for the base 5 to find the product of 7.5 by 2.6.

$$\begin{aligned} 7.5 \times 2.6 &\stackrel{a}{=} 5^{1.25} \times 5^{0.59} \\ & 5^{1.84} \\ &\stackrel{a}{=} 19.3 \end{aligned}$$

Actually, $7.5 \times 2.6 = 19.5$. Now, if you had available a larger-scale drawing of the exponential curve for the base 5, you could carry out the computation as follows:

F. The table referred to is actually a four-place table of common logarithms, except that it is headed:

Coordinates of points in the exponential curve for the base 10.

$$y = 10^x$$

and the left-hand column on each page, which is usually labelled 'N', is labelled 'y'. Again, do not refer to this table as 'a table of logarithms'.

$$7.5 \times 2.6 \stackrel{a}{=} 5^{1.2519} \times 5^{0.5937}$$
$$= 5^{1.8456}$$
$$\stackrel{a}{=} 19.50$$

A picture of an exponential curve for the base 5 would have to be quite large to permit you to obtain $5^{1.2519}$ as an approximation to 75. Even a table which listed the coordinates of points on that curve would have to be quite long. Probably no one has ever made such a drawing or such a table.

Fortunately, however, mathematicians have constructed very extensive (and accurate) tables of coordinates for points contained in <u>the exponential curve for the base 10</u>. Since, as you have seen in Parts D and E, the base of the exponential curve does not affect the computational methods we can use such a table to improve the accuracy of our answers. We give a table of coordinates of points in the exponential curve for the base 10 on the last three pages of this unit. The table lists the ordered pairs corresponding to points on or very close to a point on the locus of '$y = 10^x$' for 900 values of 'x' from 0.0000 to 0.9996.

Here is how you read the table. Suppose you want to find an approximation to the abscissa of that point on the locus of '$y = 10^x$' whose ordinate is 6.23. In the columns headed 'y' search for '6.2'. You will find '6.2' on the second page of the table. Place your right index finger on '6.2' and move your hand to the right until your finger is touching the numeral '.7945' which is in the column headed '3'. Now you know that the point (6.23, 0.7945) is very close to a point on the exponential curve. In other words,

$$6.23 \stackrel{a}{=} 10^{0.7945}$$

Study the table until you understand how to use it. To check your understanding, see if you obtain the following results from the table.

$$2.00 \stackrel{a}{=} 10^{0.3010}$$

$$2.01 \stackrel{a}{=} 10^{0.3032}$$

$$2.02 \stackrel{a}{=} 10^{0.3054}$$

$$4.37 \stackrel{a}{=} 10^{0.6405}$$

$$10^{0.9390} \stackrel{a}{=} 8.69$$

$$10^{0.0128} \stackrel{a}{=} 1.03$$

* * *

Use the table to solve the following equations.

Sample 1. $10^{3.5527} = k$

Solution. The exponent 3.5527 is not listed among the abscissas in the table. However, we know that $10^{3.5527} = 10^{3} \times 10^{0.5527}$ and the exponent 0.5527 is listed in the table. Since

$$10^{0.5527} \stackrel{a}{=} 3.57$$

then

$$10^{3.5527} \stackrel{a}{=} 3.57 \times 10^{3} = 3570 .$$

The root of the given equation is approximately 3570.

Sample 2. $853 = 10^{m}$

Solution. The power 853 is not listed among the ordinates in the table. Since

$$853 = 8.53 \times 10^{2}$$

and since

$$8.53 \stackrel{a}{=} 10^{0.9309}$$

then

$$853 \stackrel{a}{=} 10^{0.9309} \times 10^{2} = 10^{2.9309}.$$

So the root of the given equation is approximately 2.9309.

F. (cont.)

1.	2.50	2.	25.0
3.	25,000	4.	.9284
5.	2.9284	6.	4.9284
7.	9.95	8.	5.06
9.	2.1335	10.	3.6395
11.	2.0414	12.	2.0043
13.	3.0334	14.	3.2553

* * *

Note, in connection with Sample 3 and Sample 4, that we do not now expect students to use interpolation in finding values not listed in the table. The process of linear interpolation is explained on pages 2-79 and 2-80 and students are told on page 2-81 that they are to use this process, whenever appropriate, from then on.

1. $10^{0.3979} = a$
2. $10^{1.3979} = b$
3. $10^{4.3979} = c$
4. $8.48 = 10^{d}$
5. $848 = 10^{u}$
6. $84800 = 10^{v}$
7. $10^{0.9978} = a$
8. $10^{0.7042} = b$
9. $136 = 10^{g}$
10. $4360 = 10^{k}$
11. $110 = 10^{k}$
12. $101 = 10^{m}$
13. $1080 = 10^{n}$
14. $1800 = 10^{p}$

<u>Sample 3</u>. $10^{0.7453} = q$

<u>Solution</u>. The exponent 0.7453 is not listed among the abscissas in the table. However, 0.7451 is listed. Since

$$10^{0.7453} \overset{a}{=} 10^{0.7451}$$

then

$$10^{0.7453} \overset{a}{=} 5.56.$$

So, the root of the given equation is approximately 5.56.

<u>Sample 4</u>. $9.5492 = 10^{w}$

<u>Solution</u>. The power 9.5492 is not listed among the ordinates. We note that

$$9.5492 \overset{a}{=} 9.55.$$

Therefore, since

$$9.55 \overset{a}{=} 10^{0.9800}$$

we can say that

$$9.5492 \overset{a}{=} 10^{0.9800}$$

and conclude that 0.9800 is an approximation to the root of the given equation.

`-75]

15.	4.27	16.	5.77
17.	10.1	18.	309
19.	3	20.	0.8035
21.	0.6355	22.	2.7177
23.	6.8048	24.	6

15. $10^{0.6308} = t$ 16. $10^{0.7610} = s$

17. $10^{1.0056} = r$ 18. $10^{2.4899} = u$

19. $10^{0.4769} = v$ 20. $6.3592 = 10^{z}$

21. $4.3152 = 10^{a}$ 22. $10^{k} = 521.62$

23. $6382419 = 10^{b}$ 24. $10^{c} = 1000010$

G. Use the table for the exponential curve for the base 10 to make computations easier in the following problems.

Sample 1. Find the area of a rectangle which is 5.64 feet wide and 18.32 feet long.

Solution. We need to find the product of 5.64 by 18.32.

$$\begin{aligned} 5.64 \times 18.32 &\overset{a}{=} 5.64 \times 18.3 \\ &= 5.64 \times 1.83 \times 10^{1} \\ &\overset{a}{=} 10^{0.7513} \times 10^{0.2625} \times 10^{1} \\ &= 10^{(0.7513 + 0.2625 + 1)} \\ &= 10^{2.0138} \\ &= 10^{0.0138} \times 10^{2} \\ &\overset{a}{=} 1.03 \times 10^{2} \\ &= 103 \end{aligned}$$

The area is approximately 103 square feet.

Sample 2. Find the area of the circle the length of whose radius is 6.08 inches.

Solution. We need to simplify the expression '$\pi(6.08)^2$'.

1. (a) 9.19×10^4 square feet

   (b) 1.57×10^2 square inches

   (c) 1.72×10^6 square miles

   (d) 9.12 square yards

   (e) 1.80×10^6 square meters

2. (a) 5.22×10^3 square inches

   (b) 11.6 square feet

   (c) 3.78×10^3 square centimeters

   (d) 5.56×10^{43} square miles

3. (a) 25.3 inches (b) 12.1 feet

   (c) 1.95×10^2 centimeters (d) 2.64×10^{22} miles

$$\begin{aligned}\pi(6.08)^2 &\stackrel{a}{=} 3.14 \times (6.08)^2\\ &\stackrel{a}{=} 10^{0.4969} \times \left(10^{0.7839}\right)^2\\ &= 10^{0.4969} \times 10^{1.5678}\\ &= 10^{2.0647}\\ &= 10^{0.0647} \times 10^2\\ &\stackrel{a}{=} 1.16 \times 10^2\\ &= 116\end{aligned}$$

The area is approximately 116 square inches.

1. Find the area of each of the rectangles whose dimensions are:

   (a) 326 feet and 282 feet

   (b) 18.7 inches and 8.40 inches

   (c) 184000 miles and 9.371 miles

   (d) 3.01 yards and 3.03 yards

   (e) 1342 meters and 1342 meters

2. Find the area of a circle the length of whose radius is:

   (a) 40.2 inches (b) 1.92 feet

   (c) 34.7 centimeters (d) 4.21×10^{21} miles

3. Find the circumference of each of the circles described in Exercise 2.

H. The table for the exponential curve for the base 10 lists coordinates of points in the first quadrant only. But just as it is possible to use the table to find abscissas for points whose ordinates are greater than 9.99, it is also possible to use the table to find coordinates of points in the second quadrant, that is, points with negative abscissas.

H. 1. 6.72×10^{-2} 2. -1.2660

3. 1.47×10^{-5} 4. -2.0159

5. 9.90×10^{-7} 6. -5.9586

7. 3.97×10^{-3} 8. 0.306

9. -5 10. -13.5031

11. no solution 12. no solution

Sample 2. Solve the equation:

$$10^{t} = 0.000397.$$

Solution. We note, first of all, that since $0.000397 < 1$, the abscissa corresponding to that point whose ordinate is 0.000397 is a negative number. Now,

$$\begin{aligned} 0.000397 &= 3.97 \times 10^{-4} \\ &\stackrel{a}{=} 10^{0.5988} \times 10^{-4} \\ &= 10^{0.5988 - 4} \\ &= 10^{-3.4012} \end{aligned}$$

So, the root of the given equation is approximately -3.4012.

[Note: You will find in later computational work that the expression '0.5988 - 4' is more useful than '-3.4012' even though it is not as simple-looking.]

Solve the following equations. Find a decimal approximation to each root.

1. $10^{-1.1729} = y$
2. $0.0542 = 10^{x}$
3. $10^{-4.8331} = t$
4. $0.00964 = 10^{s}$
5. $10^{-6.0044} = u$
6. $0.00000110 = 10^{v}$
7. $10^{-2.4016} = k$
8. $10^{-0.5142} = k$
9. $10^{a} = 0.00001$
10. $10^{b} = 0.0314 \times 10^{-12}$
11. $10^{c} = -100$
12. $10^{d} = -0.1931$

I. For each of the following write a scientific numeral which stands for a number which is approximately equal to the given number.

4

Answer to 'Why this step?': If the name of the exponent is written in the form:

$$\text{'}0 \cdot --- + n\text{'},$$

in which 'n' stands in the place of a name for an integer, then that integer-numeral is the exponent symbol in the exponential which makes up part of the scientific name of the power.

1. 1.53×10^{-2}
2. 4.25×10^{-5}
3. 3.06×10^{-1}
4. 1.27×10^{-3}
5. 1.72×10^{-9}
6. 4.93×10^{-5}

Sample. 0.0312×0.000846

Solution. $0.0312 \times 0.000846 = (3.12 \times 10^{-2}) \times (8.46 \times 10^{-4})$

$= 3.12 \times 8.46 \times 10^{-6}$

$\stackrel{a}{=} 10^{0.4942} \times 10^{0.9274} \times 10^{-6}$

$= 10^{1.4216 - 6}$

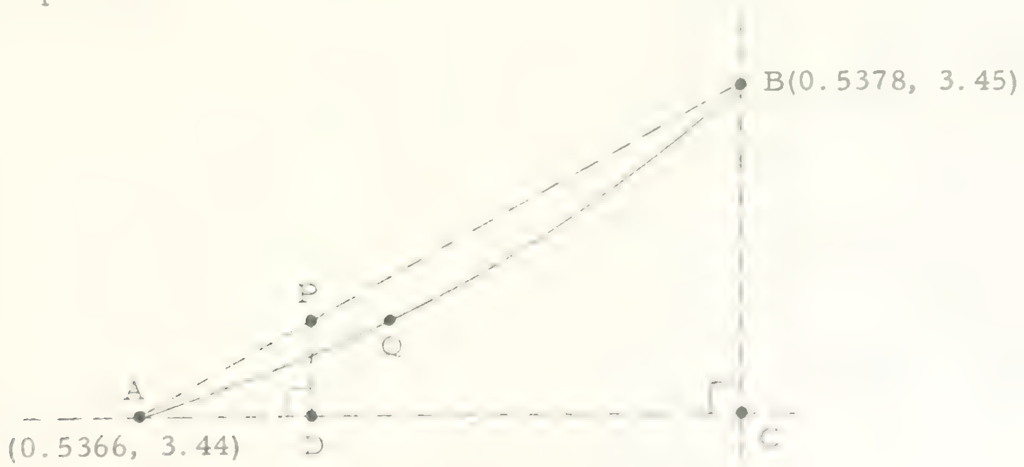
$= 10^{0.4216 - 5}$ [Why this step?]

$\stackrel{a}{=} 2.64 \times 10^{-5}$

1. 0.00781×1.92
2. 0.000624×0.0681
3. 0.0000678×4510
4. $0.00362 \times 0.0182 \times 19.3$
5. $5.26 \times 10^{-6} \times 3.27 \times 10^{-4}$
6. $1.11 \times 10^{20} \times 4.44 \times 10^{-25}$

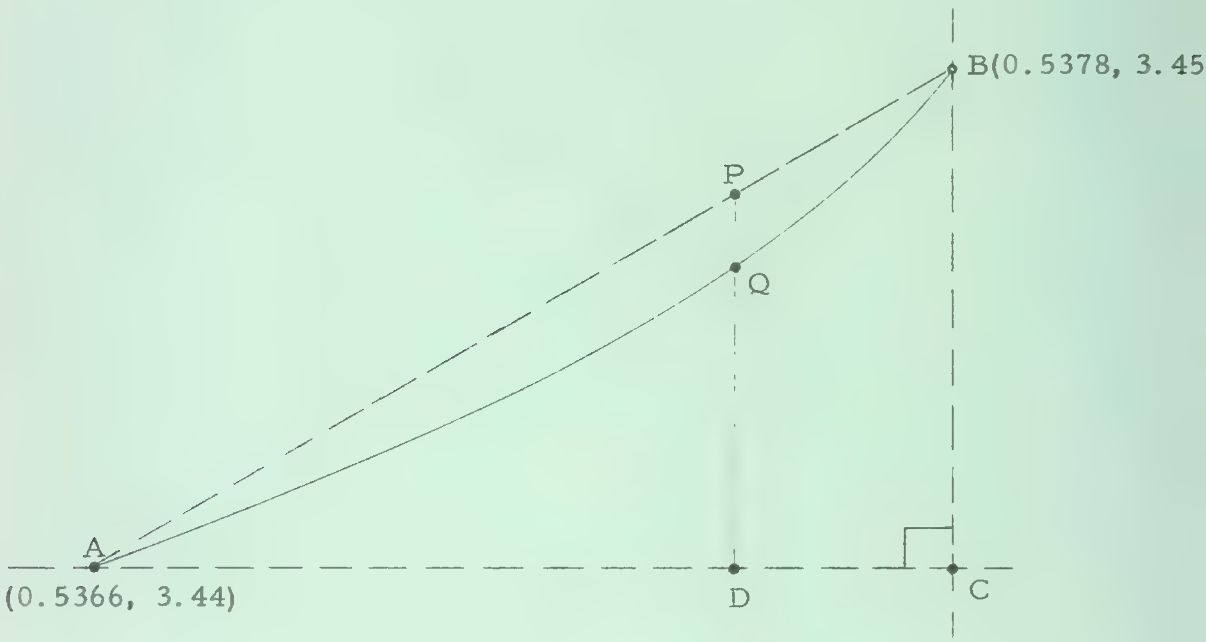
LINEAR INTERPOLATION

We can obtain approximations to the coordinates of more points than are listed in the table by a method called linear interpolation. If you select two points in an exponential curve which are sufficiently close together, the chord ("linear" segment) which joins these points lies very close to the curve. [This is a consequence of characteristic (ii) of exponential curves. See page 2-61.] Here is an enlarged picture of a small piece of the exponential curve for the base 10.



EXERCISES

A. In connection with the solution of the sample students should, as suggested on page 2-81, make a sketch like the following.



$$\frac{m(\overline{PD})}{m(\overline{BC})} = \frac{m(\overline{AD})}{m(\overline{AC})}$$

(So if $x = m(\overline{PD})$ then the ordinate of P is $3.44 + x$ and this is also, approximately, the ordinate of Q.)

* * *

Students should make a sketch like the above, or like that on page 2-79, whichever is appropriate, for each interpolation problem until they are certain that they understand the process of linear interpolation.

Suppose we want to find the abscissa of the point Q on the curve whose ordinate is 3.443. This ordinate is not listed in the table. However, the coordinates of points A and B are listed. Now, if P is the point on the chord $\overline{AB}$ whose ordinate is 3.443, the same as that of Q, then the abscissa of P is an approximation to that of Q. By similar triangles it is easy to find the abscissa of P and, hence, to approximate the abscissa of Q. Since $\Delta APD \sim \Delta ABC$, we have:

$$\frac{m(\overline{AD})}{m(\overline{AC})} = \frac{m(\overline{PD})}{m(\overline{BC})},$$

or:

$$\begin{aligned} m(\overline{AD}) &= \frac{3.443 - 3.44}{3.45 - 3.44} \times (0.5378 - 0.5366) \\ &= \frac{0.003}{0.01} \times (0.0012) \\ &= 0.3(0.0012) \\ &\overset{a}{=} 0.0004 \end{aligned}$$

Therefore, the abscissa of P is approximately

$$0.5366 + 0.0004$$

or

$$0.5370$$

So, the abscissa of Q is approximately 0.5370.

EXERCISES

A. Use linear interpolation to approximate the missing coordinate for each of the points in the exponential curve for the base 10.

Sample. (0.5374,)

Solution. No point Q with this abscissa is listed in the table. However, it is clear that the point in question lies between the listed points A and B.

A: (0.5366, 3.44)

0.0008 x

Q: (0.5374,) 0.01

0.0012

B: (0.5378, 3.45)

A. (cont.)

1.	0.8053	2.	6.982
3.	0.5420	4.	9.852
5.	0.8924	6.	6.693
7.	1.7658	8.	32.52
9.	0.8832 - 1	10.	0.4243

From similar triangles,

$$\frac{x}{0.01} \stackrel{a}{=} \frac{0.0008}{0.0012},$$

so

$$x \stackrel{a}{=} 0.007.$$

Therefore, the ordinate of Q is approximately 3.447. [You should make a diagram similar to the one on page 2-79 and interpret the above procedure geometrically.]

1. (, 6.387)	2. (0.8440,)
3. (, 3.483)	4. (0.9935,)
5. (, 7.805)	6. (0.8256,)
7. (, 58.32)	8. (1.5121,)
9. (, 0.7642)	10. (-0.3723,)

> Note: Use linear interpolation henceforth in this unit when you are looking for coordinates of points (in the exponential curve for the base 10) which are not listed in the table.

B. Use the table of coordinates for the exponential curve for the base 10 to reduce the computational work in finding approximate answers to the following problems.

Sample. Find the area of the triangle whose base is 0.00652 feet long and whose height is 0.0532 feet.

[2-82]

B. 1. 4.439×10^{-4} sq. ft. 2. 2.318×10^{-3} cu. M.

3. 1.751×10^{-3} cu. M. 4. 3.182×10^{-6} cu. ft.

5. 1.046×10^{-3} sq. ft. 6. 1.937×10^{-1} cu. ft.

7. 1.411 sq. ft. 8. 1.885 sq. ft.

9. 1.313×10^{-8} sq. km. 10. 1.395×10^{-7} cu. cm.

Solution. $\frac{1}{2} \times 0.00652 \times 0.0532$

$= (5 \times 10^{-1}) \times (6.52 \times 10^{-3}) \times (5.32 \times 10^{-2})$

$= 5 \times 6.52 \times 5.32 \times 10^{-6}$

$\stackrel{a}{=} 10^{0.6990} \times 10^{0.8142} \times 10^{0.7259} \times 10^{-6}$

$= 10^{2.2391 - 6}$

$= 10^{0.2391 - 4}$

$\stackrel{a}{=} 1.73 \times 10^{-4}$

The area of the triangle is approximately 1.73×10^{-4} square feet (or 0.000173 square feet).

1. Find the area of the parallelogram which has a base 0.005624 feet long and a height of 0.07893 feet.
2. Find the volume of the rectangular solid whose dimensions are 0.7802 meters, 0.009905 meters, and 0.301 meters.
3. Find the volume of the cone whose base has a radius of 0.0986 meters and whose height is 0.172 meters. [Recall: $V = \frac{1}{3}\pi r^2 h$.]
4. Find the volume of the sphere the length of whose radius is 0.009124 feet. [Recall: $V = \frac{4}{3}\pi r^3$.]
5. Find the surface area of the sphere in Exercise 4. [Recall: $A = 4\pi r^2$.]
6. Find the volume of the cylinder which has a base of radius 0.2745 feet and a height of 0.8183 feet. [Recall: $V = \pi r^2 h$.]
7. Find the lateral area of the cylinder in Exercise 6. [Recall: $L = 2\pi rh$.]
8. Find the total area of the cylinder in Exercise 6. [Recall: $T = 2\pi rh + 2\pi r^2$, or: $T = 2\pi r(h + r)$.]
9. Find the area of the trapezoid which has bases 0.00003768 and 0.000005892 kilometers in length, and a height of 0.0006014 kilometers. [Recall: $A = \frac{1}{2}h(B + b)$.]
10. Find the volume of a cube which has an edge of length 0.005185 centimeters.

C. The exercises in the preceding part were basically multiplication problems. You can use the exponential curve for the base 10 to reduce the computational work in dividing and in simplifying exponentials.

Sample 1. Simplify: $8790 \div 324$.

Solution.

$$\frac{8790}{324} = \frac{8.790 \times 10^3}{3.24 \times 10^2}$$

$$\stackrel{a}{=} \frac{10^{0.9440} \times 10^3}{10^{0.5105} \times 10^2}$$

$$= 10^{0.4335} \times 10^1$$

$$\stackrel{a}{=} 2.71 \times 10^1, \text{ or } 27.1$$

Sample 2. Simplify: $421 \div 89600$

Solution.

$$\frac{421}{89600} = \frac{4.21 \times 10^2}{8.96 \times 10^4}$$

$$\stackrel{a}{=} \frac{10^{0.6243} \times 10^2}{10^{0.9523} \times 10^4}$$

[Note that if we subtract 0.9523 from 0.6243 we shall obtain a negative exponent which will have to be treated as in Sample 1 of Part H. To avoid this bit of work we proceed as follows:]

$$\frac{10^{1.6243} \times 10^1}{10^{0.9523} \times 10^4}$$

$$10^{0.6720} \times 10^{-3}$$

$$= 4.70 \times 10^{-3} \text{ (or 0.00470)}$$

<u>Sample</u> <u>3</u>. Simplify: $\sqrt[5]{962}$

<u>Solution</u>.

$$\begin{aligned}
\sqrt[5]{962} &= (962)^{\frac{1}{5}} \\
&= (9.62 \times 10^{2})^{\frac{1}{5}} \\
&\stackrel{a}{=} (10^{0.9832} \times 10^{2})^{\frac{1}{5}} \\
&= (10^{2.9832})^{\frac{1}{5}} \\
&= 10^{(2.9832 \times \frac{1}{5})} \\
&= 10^{0.59664} \\
&\stackrel{a}{=} 3.95
\end{aligned}$$

<u>Sample</u> <u>4</u>. Simplify: $\sqrt[3]{0.000962}$

<u>Solution</u>.

$$\begin{aligned}
\sqrt[3]{0.000962} &= (0.000962)^{\frac{1}{3}} \\
&= (9.62 \times 10^{-4})^{\frac{1}{3}} \\
&\stackrel{a}{=} (10^{0.9832} \times 10^{-4})^{\frac{1}{3}}
\end{aligned}$$

[We could write as the next expression:

$$(10^{0.3274} \times 10^{-1.3333}).$$

On adding exponents we would obtain a negative exponent and proceed as in Sample 1 of Part H. We can avoid this bit of work by using the following trick:]

$$(10^{2.9832} \times 10^{-6})^{\frac{1}{3}}$$

$$10^{0.9944} \times 10^{-2}$$

$$\stackrel{a}{=} 9.87 \times 10^{-2}, \text{ or } 0.0987$$

they may not feel certain that the equation '$2^x = 16$' has only one root. The situation is more serious in the case of:

$$0.3010 \stackrel{a}{=} \text{the logarithm to the base 10 of 2.}$$

A student may admit that $10^{0.3010} \stackrel{a}{=} 2$, but not be certain that there is any number x such that $10^x = 2$, exactly. And again, granting that there is one such x, can he be sure that there is only one?

If you are lucky enough to have a student who asks such questions at this point, tell him that these questions are eminently proper. Logically, before using the phrase 'the logarithm to the base 5 of 25' one should prove that there is just one number x such that $5^x = 25$. On pages 2-88 through 2-91 we do what we can at this stage to prove that this is so. Pedagogically it seems better to introduce the word 'logarithm' first and then justify our use of it after the student has had a chance to become accustomed to the word.

C. 1.	8.447×10^{-2}	2.	1.843×10^{-5}
3.	2.893×10^{-6}	4.	1.542×10^{2}
5.	2.42×10^{-1}	6.	9.862×10^{0}
7	1.77	8.	3.184×10^{-1}
9.	1.013×10^{-1}	10.	1.46
11.	1.33	12.	1.23
13.	1.65	14.	1.40
15.	1.28	16.	1.22
17.	1.005×10^{1}	18.	19.0
19.	.684	20.	.641

* * *

The remaining sections of this unit, 2.08 and 2.09, deal with logarithms. Students are first shown, by examples, the meaning of the word 'logarithm', i.e., that, for all x, b, and y, x is the logarithm to the base b of y just if $b^x = y$. [Note that we say 'x is the logarithm to the base b of y', parodying the notation '$\log_b y$', rather than, as some say, 'x is the logarithm of y to the base b'.]

Students who have thoroughly understood our earlier insistence that the use of phrases of the form 'the --- such that ------' requires a preliminary proof that there is one and only one --- such that ------ may boggle at such a statement as:

4 = the logarithm to the base 2 of 16.

Although they may grant that $2^4 = 16$ (there is one x such that $2^x = 16$),

(continued on T. C. 85B)

Use the table for the exponential curve for the base 10 to find approximations (in decimal numeral or scientific numeral form) for the numbers listed below.

1. $310 \div 3670$	2. $0.07316 \div 3962$
3. $(0.01425)^3$	4. $0.00631 \times 1760 \div 0.0720$
5. $(0.01425)^{\frac{1}{3}}$	6. $(3.141)^2$
7. $(3.141)^{\frac{1}{2}}$	8. $1 \div 3.141$
9. $1 \div 9.872$	10. $\sqrt[3]{3.141}$
11. $\sqrt[4]{3.141}$	12. $\sqrt[5]{3.141}$
13. $\sqrt{2.726}$	14. $\sqrt[3]{2.726}$
15. $\sqrt[4]{2.726}$	16. $\sqrt[5]{2.726}$
17. $\sqrt[7]{1758} \times \sqrt[6]{1707}$	18. $\sqrt[4]{0.09705} \div \sqrt[5]{0.006908}$
19. $\sqrt[5]{17.23} \div \sqrt[3]{17.23}$	20. $\sqrt{0.06912} \div \sqrt[3]{0.06912}$

2.08 Logarithms.--The computations you have carried out in the preceding exercises with the help of exponential curves are commonly called computations with logarithms. A logarithm is an exponent, the abscissa of a point on an exponential curve. For example, since the point (2, 25) is a point on the exponential curve for the base 5 [because $5^2 = 25$], the number 2 is often called

the logarithm to the base 5 of 25.

Similarly,

4 = the logarithm to the base 2 of 16 $[2^4 = 16]$

1 = the logarithm to the base 7 of 7 $[7^1 = 7]$

3 = the logarithm to the base 4 of 64 [Why?]

$0.3010 \stackrel{a}{=}$ the logarithm to the base 10 of 2 $[10^{0.3010} \stackrel{a}{=} 2]$

0 = the logarithm to the base π of 1 $[\pi^0 = 1]$

-4 = the logarithm to the base $\frac{1}{2}$ of 16 $[(\frac{1}{2})^{-4} = 16]$

'2-86]

B. 1. $10^2 = 100$

2. $5^{0.78} \stackrel{a}{=} 3.5$

3. $5^3 = 125$

4. $x^{2.3} = 17$

5. $5^{18} = y$

6. $10^{0.0755} \stackrel{a}{=} 1.19$

7. $10^x = 13$

8. $5^{-2} = 0.04$

9. $x^{\frac{1}{2}} = 5$

10. $96^0 = 1$

11. $3^y = 26$

12. $\left(\frac{1}{5}\right)^{-0.85} \stackrel{a}{=} 3.89$

13. $10^m = 125.6$

14. $9^x = 13$

[2-80]

A. 1. 3 = the logarithm to the base 5 of 125

2. 3 = the logarithm to the base 6 of 216

3. 1 = the logarithm to the base 3 of 3

4. $0.477 \stackrel{a}{=}$ the logarithm to the base 10 of 3

5. 10 = the logarithm to the base 2 of 1024

6. $0.0780942 \stackrel{a}{=}$ the logarithm to the base 10 of 1.1970

7. 0 = the logarithm to the base 5 of 1

8. -2 = the logarithm to the base 5 of 0.04

9. -3 = the logarithm to the base 10 of 0.001

10. a = the logarithm to the base 10 of 50

11. a = the logarithm to the base 3 of b

12. b = the logarithm to the base a of c

[It may be worth-while to set variations of Exercises 7 and 9 as a basis for generalizations which, if verbalized, might read:

For every $b > 0$, 0 is the logarithm to the base b of 1

and:

For every integer n, n is the logarithm to the base 10 of 10^n.]

13. $2 = \log_9 81$

14. $4 \stackrel{a}{=} \log_{5.5} 910$

15. $3 \stackrel{a}{=} \log_{13.1} 2250$

16. $-2 \stackrel{a}{=} \log_7 0.0204$

17. $x = \log_7 104$

18. $5 = \log_x 5.12$

19. $9.06 = \log_4 x$

20. $x = \log_a 328$

21. $10 = \log_a b$

* * *

Correction: In Exercises 6, 7, and 13 of Part B the base is 10.

* * *

(continued on T. C. 86B)

EXERCISES

A. Translate each of the following equations into one which contains the word 'logarithm':

1. $5^3 = 125$	2. $6^3 = 216$	3. $3^1 = 3$
4. $10^{0.477} \stackrel{a}{=} 3$	5. $2^{10} = 1024$	6. $10^{0.0780942} \stackrel{a}{=} 1.1970$
7. $5^0 = 1$	8. $5^{-2} = 0.04$	9. $10^{-3} = 0.001$
10. $10^a = 50$	11. $3^a = b$	12. $a^b = c$

Note: Instead of translating '$3^5 = 243$' into '5 is the logarithm to the base 3 of 243' we can abbreviate by writing '$5 = \log_3 243$'. Use the abbreviated translation for each of the following expressions.

13. $9^2 = 81$	14. $5.5^4 \stackrel{a}{=} 910$	15. $13.1^3 \stackrel{a}{=} 2250$
16. $7^{-2} \stackrel{a}{=} 0.0204$	17. $7^x = 104$	18. $x^5 = 5.12$
19. $4^{9.06} = x$	20. $a^x = 328$	21. $a^{10} = b$

B. Translate each of the following equations into one which contains an exponential:

1. $\log_{10} 100 = 2$	2. $\log_5 3.5 \stackrel{a}{=} 0.78$
3. $\log_5 125 = 3$	4. $\log_x 17 = 2.3$
5. $\log_5 y = 13$	6. $\log 1.19 \stackrel{a}{=} 0.0755$
7. $\log 13 = x$	8. $\log_5 0.04 = -2$
9. $\log_x 5 = 1$	10. $\log_{96} 1 = 0$
11. $\log_3 26 = y$	12. $\log_{1} 3.89 \stackrel{a}{=} -0.85$
13. $\log 125.6 = m$	14. $\log_{6} 13 = x$

C. Logarithms to the base 10 are frequently called common logarithms. The statement:

$$2 = \log_{10} 100$$

is sometimes read:

2 is the common logarithm of 100

and is sometimes abbreviated:

$$2 = \log 100.$$

Solve the following equations.

Sample 1. $\log 59 = k$

Solution. This equation is equivalent to:

$$10^k = 59.$$

Use the table of coordinates for the exponential curve for the base 10 (frequently called a table of common logarithms). Since

$$59 = 5.9 \times 10^1$$

then

$$\begin{aligned} 59 &\stackrel{a}{=} 10^{0.7709} \times 10^1 \\ &= 10^{1.7709} \end{aligned}$$

Therefore,

$$\log 59 \stackrel{a}{=} 1.7709.$$

So, the root of 'log 59 = k' is approximately 1.7709.

Sample 2. $\log t = -2.7$

Solution. We seek the root of:

$$10^{-2.7} = t$$

Since

$$\begin{aligned} 10^{-2.7} &= 10^{0.3-3} \\ &= 10^{0.3} \times 10^{-3} \\ &\stackrel{a}{=} 2 \times 10^{-3} \\ &= 0.002, \end{aligned}$$

the root of 'log t = -2.7' is approximately 0.002.

C. 1. 0.4771 | 2. 0.5384

C. 1.	0.4771	2.	0.5384
3.	1.5384	4.	5.5384
5.	0.5384 - 3	6.	0.7910 - 2
7.	2.506×10^{2}	8.	6.897
9.	3.61×10^{-2}	10.	4.59×10^{-2}
11.	0.0282 - 6	12.	0.7751 - 1
13.	3	14.	1

* * *

We take up here the questions discussed on T. C. 85:

> Given numbers b and y can there be more than one number x such that $b^x = y$?

and:

> Given numbers b and y must there be a number x such that $b^x = y$?

1. $\log 3 = s$
2. $\log 3.455 = r$
3. $\log 34.55 = m$
4. $\log 345500 = n$
5. $\log 0.003455 = p$
6. $\log 0.0618 = q$
7. $\log a = 2.3990$
8. $\log b = 0.8386$
9. $\log c = -1.4425$
10. $\log d = 3.6618 - 5$
11. $\log 0.000001067 = f$
12. $\log 0.5959 = g$
13. $\log 1000 = h$
14. $\log t = 0$

RULES FOR LOGARITHMS

You have seen that, for example, '2 is the logarithm to the base 5 of 25' is another way of saying that 25 is the power with base 5 and exponent 2:

$$\text{'}2 = \log_5 25\text{' is equivalent to '}5^2 = 25\text{'}.$$

Similarly, since $5^0 = 1$, $\log_5 1 = 0$, and, since $5^{-1} = 0.2$, $\log_5 0.2 = 1$. There is one thing which we must check before going further: We know, for example, that $5^{\frac{1}{2}} = \sqrt{5}$, and so we would agree that $\log_5 \sqrt{5} = \frac{1}{2}$.

But perhaps there exists a number $a \neq \frac{1}{2}$ such that $5^a = \sqrt{5}$. If so we would be equally inclined to agree that $\log_5 \sqrt{5} \neq \frac{1}{2}$. [Why should the possibility of this be disturbing?]

Before reading further, look at the picture, on page 2-65, of the exponential curve for the base 5 and see if you have reason to think that the equation '$5^x = \sqrt{5}$' has at most one root.

As suggested above, if we are to avoid inconsistencies in defining logarithms to the base 5, we need to know that there do not exist two real numbers a and b such that $5^a = 5^b$. [Why would knowing this convince us that no number can have two logarithms to the base 5?] In order to prove that for every a and b, if $5^a = 5^b$ then $a = b$, let us consider the equation:

$$5^a = 5^b.$$

Since, for every number r, $5^r \neq 0$, this equation is equivalent to:

$$5^a \cdot 5^{-b} = 5^b \cdot 5^{-b} \qquad \text{[Why?].}$$

That is, it is equivalent to:

$$5^{a-b} = 1 \qquad \text{[Why?].}$$

Hence, in order to prove:

For every a and b, $5^a = 5^b$ only if $a = b$,

it is sufficient to prove:

(*) For every x, $5^x = 1$ only if $x = 0$ [Why?].

Now, if $x \neq 0$ and $5^x = 1$ then [by characteristic (i) of exponential curves, page 2-61]

$$\left(5^x\right)^{\frac{1}{x}} = 1^{\frac{1}{x}},$$

or, since $1^{\frac{1}{x}} = 1$,

$$5 = 1.$$

Since $5 \neq 1$, it follows that there is no number $x \neq 0$ such that $5^x = 1$. Hence, if $5^x = 1$ then $x = 0$. Since we have proved (*), there do not exist two numbers a and b such that $5^a = 5^b$. Consequently, every number has at most one logarithm to the base 5.

We come now to a second thing which must be checked before we make much use of logarithms: What numbers have logarithms to, say, the base 5? For example, do you think that, for every real number y, there is at least one real number x such that $5^x = y$?

You know that, for every real number x, $5^x \neq 0$, and that $5^x = \left(5^{x/2}\right)^2 \geq 0$. Hence, if there exists an x such that $5^x = y$, $y > 0$. In other words, a number which has a logarithm to the base 5 must be positive. This suggests the question: Does every positive number have a logarithm to the base 5? Look now at the drawing on page 2-65 and guess the answer to this question.

In more advanced mathematics courses it is proved that

for every $y > 0$ there is an x such that $5^x = y$.

[2-90]

of

A neater treatment if one wants to take care at once of each base $a > 1$ is to use the fact that, if $a > 1$ and $n > \frac{y - 1}{a - 1}$ then $y < a^n$. This is a consequence of Bernovilli's Inequality, for which see the Review Exercises.

Once one has proved that, for every $a > 1$ and every $y > 0$, there is an x such that $a^x = y$, one can handle the case in which $0 < a < 1$ as follows: If $0 < a < 1$ then $\frac{1}{a} > 1$, so, as previously proved, for every $y > 0$ there is an x such that $\left(\frac{1}{a}\right)^x = y$. But then $a^{-x} = y$ (i.e., there is a number z such that $a^z = y$).]

of

To prove that, for every positive integer n, $n < 5^n$, requires the use of mathematical induction. The property in question is expressed by:

$$\ldots < 5^{\cdots} .$$

Since $1 < 5$, 1 has the property. Suppose then, that, for a given positive integer k,

$$k < 5^k .$$

Then $$k + 1 < 5^k + 1 .$$

But $$5^k + 1 < 5^k + 5^k \cdot 4 \qquad [\text{Since } 1 < 5^k \cdot 4]$$
$$= 5^{k+1} .$$

Hence, for every positive integer k, if $k < 5^k$ then $k + 1 < 5^{k+1}$, i.e., the property is hereditary. As a consequence of the Principle of Mathematical Induction for positive integers, for every positive integer n, $n < 5^n$.

[The following material suggests how cases in which the base $\neq 5$ might be handled. You probably will not need it in your teaching. The inequality '$n < a^n$' holds for every positive integer n and every $a \geq 1.7$ (the critical step in the above proof depended on the fact that $1 < 5^1 \cdot (5 - 1)$, and it is the case that $1 < a^1(a - 1)$ if $a \geq 1.7$). Hence the same argument which shows that, for every $y > 1$ there is a number x such that $5^x = y$ will also show, for every $a \geq 1.7$, that, for every $y > 1$ there is a number x such that $a^x = y$. Moreover, for any $a > 1$ and every sufficiently large integer n, $n < a^n$, so the argument can be made to apply to each $a > 1$ if we choose not only $\geq y$, but also large enough so that $n < a^n$ (how large we must choose this integer will depend on how close a is to 1).

(continued on T. C. 90B)

The proof depends on the following theorem, which is a consequence of characteristics (i) and (ii) of exponential curves:

> For every number y, if there exist numbers x_1 and x_2 such that
>
> $$5^{x_1} < y < 5^{x_2}$$
>
> then there is a number x such that
>
> $$5^x = y$$
>
> (and x is between x_1 and x_2).

We shall not prove this theorem. But by using it we shall prove that every positive number has a logarithm to the base 5.

We shall begin by proving this for every number $y > 1$. The theorem just quoted tells us that, for every $y > 1$, there is an x such that $5^x = y$ if, for each $y > 1$, there are numbers x_1 and x_2 such that

$$5^{x_1} < y < 5^{x_2}.$$

Since $5^0 = 1$, and $1 < y$, all that remains to be done is to show that there is a number x_2 such that

$$y < 5^{x_2}.$$

Now, it is easy to see that, for every positive integer n, $n < 5^n$. Hence, if n is the smallest integer greater than y then $y < n$ and $n < 5^n$, so

$$y < 5^n.$$

The theorem now tells us that, for every $y > 1$ there is a number $x > 0$ and $<$ the smallest integer greater than y such that

$$5^x = y.$$

On the other hand, for every y such that $0 < y < 1$, $\frac{1}{y} > 1$, so, as has just been proved, there exists a number $x > 0$ such that

$$5^x = \frac{1}{y}.$$

But this equation is equivalent to:

$$5^{-x} = y.$$

Summarizing:

> For each number $y > 0$, there is just one number x such that $5^x = y$. If $0 < y < 1$ then the x such that $5^x = y$ is < 0. The x such that $5^x = 1$ is 0. If $1 < y$ then the x such that $5^x = y$ is > 0.

Or:

> Every positive number has a unique logarithm to the base 5. For every $y > 0$, if $y < 1$ then $\log_5 y < 0$; $\log_5 1 = 0$; if $y > 1$ then $\log_5 y > 0$.

In the same way one can show, for each real number $a > 1$, that every positive real number has a logarithm to the base a, and that, for every $y > 0$, $\log_a y < 0$ if $y < 1$; $\log_a 1 = 0$; $\log_a y > 0$ if $y > 1$. [A similar result holds for every real number a such that $0 < a < 1$. State it! What can you say about the possibility of defining logarithms to the base 1?]

We are now justified in stating the following defining principle:

> For every real number a such that $0 < a \neq 1$, and every real number $y > 0$, $\log_a y$ is the number x such that $a^x = y$.

Before stating it we had to be sure that if $0 < a \neq 1$ and $y > 0$ then (1) there is a number x such that $a^x = y$, and (2) there are not two numbers x such that $a^x = y$. If either (1) or (2) were not the case for some number a and some number y, then it would be nonsense to speak of the number x such that $a^x = y$.

An immediate consequence of the defining principle is that

> For every real number a such that $0 < a \neq 1$, and every real number $y > 0$,
>
> $$a^{\log_a y} = y.$$

[2-92]

EXERCISES

A. 1. 3 2. -2 3. 2

4. -1 5. 2 6. $\frac{1}{2}$

7. $\frac{2}{3}$ 8. $\frac{4}{3}$ 9. 2

B. $\log_a a = 1$ because $a^1 = a$.

Also,

> For every real number a such that $0 < a \neq 1$, and every real number x,
>
> $$\log_a a^x = x.$$

EXERCISES

A. Simplify.

1. $\log_5 125$
2. $\log_5 0.04$
3. $\log_3 9$
4. $\log_{0.5} 2$
5. $\log_{\frac{1}{4}} \frac{1}{16}$
6. $\log_{100} 10$
7. $\log_8 4$
8. $\log_8 16$
9. $\log_8 64$

B. Because logarithms to any base are defined as exponents of powers with that base, rules for calculating with exponents can be transformed into rules for calculating with logarithms. For example:

> For every real number a such that $0 < a \neq 1$,
>
> $$\log_a 1 = 0 \quad \text{and} \quad \log_a a = 1.$$

Proof: You know that, for every real number a, $a^0 = 1$. Hence, by the defining principle, if $0 < a \neq 1$ then $\log_a 1 = 0$. Prove the second part of the boxed theorem yourself.

Here is a second rule.

> For every real number a such that $0 < a \neq 1$, for every $x > 0$, and for every $y > 0$,
>
> $$\log_a (xy) = \log_a x + \log_a y.$$

B. (cont.)

1. 71.24 2. 127.8 3. 376.3

[It is worth a student's while to learn to outline the solution of a problem before working out the details.]

Proof: By two uses of the defining principle,

$$x \cdot y = a^{\log_a x} \cdot a^{\log_a y}.$$

By the addition rule for exponents it follows that

$$x \cdot y = a^{\log_a x + \log_a y}.$$

So, by the defining principles

$$\log_a (xy) = \log_a x + \log_a y.$$

Use the boxed theorem above and the chart on page 2-65 to simplify each of the following.

1. (5.2)(13.7)

[Note: Arrange your work in a tabular form as shown below.

$$\text{to the base 5} \left\{ \begin{array}{rl} \log(5.2)(13.7) & = \log 5.2 + \log 13.7 \\ \log 5.2 & \stackrel{a}{=} \\ (+)\ \log 13.7 & \stackrel{a}{=} \\ \log(5.2)(13.7) & \stackrel{a}{=} \\ (5.2)(13.7) & \stackrel{a}{=} \end{array} \right.$$
]

2. (21.3)(6)

3. (8.73)(43.1)

C. You can find logarithms to the base 10 by using the table of coordinates for the exponential curve for the base 10. Logarithms to the base 10 are often called <u>common logarithms</u> and a table such as that just referred to is called 'a table of common logarithms'. Such a table lists approximations to the common logarithms of certain numbers ≥ 1 and < 10. For example, referring to the table on the last three pages of this unit, the entry '.5224' in the twenty-fourth row and fifth column tells you that

$$\log_{10} 3.33 \stackrel{a}{=} 0.5224.$$

You have learned that you can use linear interpolation to find approximations to the common logarithms of other numbers ≥ 1 and < 10. For example,

$$\log_{10} 3.332 \stackrel{a}{=} 0.5227,$$

C. One answer to 'Why 'n'?' is to refer to the boxed statement at the top of page 2-92.

* * *

One answer to 'Why?' is that the mantissa of $\log_{10} x = \log_{10} x_1$ where $1 \leq x_1 < 10$.

* * *

1. 45.43
2. 8.906×10^{-2}
3. 801.8
4. 1.7925×10^{-2}
5. 0.6826
6. -8664

because 0.5227 is, approximately, "$\frac{2}{10}$ of the way from 0.5224 to 0.5237". [Explain more fully.] With the help of scientific notation you can find an approximation to the common logarithm of <u>any</u> positive number. For example, since

$$33.32 = 3.332 \times 10^1,$$

the boxed theorem at the bottom of page 2-92 tells you that

$$\log_{10} 33.32 = \log_{10} 3.332 + \log_{10} 10^1$$

$$\stackrel{a}{=} 0.5227 + 1.$$

Similarly,

$$0.003332 = 3.3332 \times 10^{-3},$$

or

$$\log_{10} 0.003332 \stackrel{a}{=} 0.5227 - 3.$$

In general, for every $x > 0$, there is an integer n and a number x_1 such that $1 \leq x_1 < 10$ and

$$x = x_1 \cdot 10^n.$$

Then

$$\log_{10} x = \log_{10} x_1 + n \qquad \text{[Why 'n'?].}$$

The number $\log_{10} x_1$ is called the <u>mantissa</u> of $\log_{10} x$, and n is called the <u>characteristic</u> of $\log_{10} x$. Evidently, for every $x > 0$,

$$0 \leq \text{ mantissa of } \log_{10} x < 1 \quad \text{[Why?];}$$

and the characteristic of $\log_{10} x$ is an integer.

Simplify each of the following using the theorems proved in Part B and the table of common logarithms in the text. For each exercise make a tabular outline as you did for the exercises in Part B.

1. (35.5)(1.28)
2. (0.0235)(3.79)
3. (1.732)(463)
4. (52.74)(0.00034)
5. (log 27)(log 3)
6. (-265)(32.7)

[Note: From now on we shall often write 'log' as an abbreviation for '$\log_{10}$'.]

* * *

C. (cont.)

7. If the characteristic of $\log_{10} x$ is negative, zero, or positive, then $0 < x < 1$, or $1 \leq x < 10$, or $10 \leq x$, respectively.

8. Since we use a base ten system of numeration, it is natural to use what we have called 'scientific numerals'. Such a numeral shows that the number it names is the product of one factor which is ≥ 1 and < 10 and a second factor which is a power of 10. The common logarithm of the original number is the sum of the common logarithm of the first factor and the exponent of the second factor.

D. 1. Since $x = \frac{x}{y} \cdot y$, $\log_a x = \log_a \left(\frac{x}{y} \cdot y\right) = \log_a \frac{x}{y} + \log_a y$. Hence, $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.

2. $\log_a \left(\frac{1}{x}\right) = \log_a 1 - \log_a x = -\log_a x$.

3. $$\frac{1}{x + \sqrt{x^2 - 1}} = \frac{1}{x + \sqrt{x^2 - 1}} \cdot \frac{x - \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = \frac{x - \sqrt{x^2 - 1}}{1}$$

$$= x - \sqrt{x^2 - 1}.$$

Hence,

$$\log_a \left(x - \sqrt{x^2 - 1}\right) = \log_a \left(\frac{1}{x + \sqrt{x^2 - 1}}\right) = -\log_a \left(x + \sqrt{x^2 - 1}\right).$$

7. What can you say of a number if the characteristic of its common logarithm is negative? Zero? Positive?

8. Why are common logarithms easier to use than logarithms to a base other than 10?

D. Here is another rule for calculating with logarithms.

> For every real number a such that $0 < a \neq 1$, for every $x > 0$, and for every $y > 0$,
>
> $$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

1. Prove the above theorem. [Hint: One proof makes use of the second boxed theorem in Part B and the fact that, for every x, and for every $y > 0$, $x = \frac{x}{y} \cdot y$.]

2. Prove the following corollary of the above theorem.

   For every real number a such that $0 < a \neq 1$, and for every $x > 0$,

   $$\log_a \left(\frac{1}{x}\right) = -\log_a x.$$

3. Use Exercise 2 to show that, for every a such that $0 < a \neq 1$, and every $x > 0$,

   $$\log_a \left(x - \sqrt{x^2 - 1}\right) = -\log_a \left(x + \sqrt{x^2 - 1}\right).$$

n.

E. [The way of exhibiting the work in the Samples is intended to suggest to students that they should <u>first</u> outline the solution (typescript) and <u>then</u> fill in the blanks (handscript).]

* * *

1.	1.717	2.	.1362
3.	7.942	4.	2.660
5.	1.253	6.	-6.264
7.	68.23	8.	.3105

E. Use a table of common logarithms to find the root of each equation.

Sample 1. $\frac{165}{3.2} = x$

Solution. $\log x = \log \frac{165}{3.2} = \log 165 - \log 3.2$

$$\log 165 \stackrel{a}{=} 2.2175$$

$$(-)\ \log 3.2 \stackrel{a}{=} \underline{0.5051}$$

$$\log \frac{165}{3.2} \stackrel{a}{=} 1.7124$$

$$\frac{165}{3.2} \stackrel{a}{=} 5.157 \times 10^{1}$$

$$= 51.57$$

An approximation to the root of the equation is 51.57.

Sample 2. $\frac{27.3}{0.024} = y$

Solution. $\log y = \log 27.3 - \log 0.024$

$$\log 27.3 \stackrel{a}{=} \overset{3}{\cancel{4}}.4363 - 2$$

$$(-)\ \log 0.024 \stackrel{a}{=} \underline{0.3802 - 2}$$

$$\log y \stackrel{a}{=} 3.0560$$

$$y \stackrel{a}{=} 1.137 \times 10^{3}$$

$$= 1,137$$

The root is approximately 1, 137.

1. $\frac{29.2}{17} = x$

2. $\frac{4.63}{34} = y$

3. $\frac{.027}{.0034} = m$

4. $125 = 47x$

5. $\frac{\log 125}{\log 47} = t$

6. $\frac{-26.5}{4.23} = x$

7. $\frac{(327)(4.8)}{23} = y$

8. $\frac{(-27)(4.2)}{-365} = x$

F. 1. The multiplication rule for exponents.

2. $x^y = \left(a^{\log_a x}\right)^y = a^{y \cdot \log_a x}$. Hence, $\log_a (x^y) = y \cdot \log_a x$.

G. 1. 6.517×10^5 2. 2.546

3. 1.717×10^{-2} 4. 0.8278

5. 8.22×10^{-5} 6. 1491

7. -2.32 8. 458.1

F. Another rule for calculating with logarithms is,

> For every real number a such that $0 < a \neq 1$, for every $x > 0$, and for every $y > 0$,
>
> $$\log_a (x^y) = y \cdot \log_a x.$$

1. The second boxed theorem in Part B is like the addition rule for exponents. What rule for exponents does the above theorem suggest?

2. Prove the above theorem. [Hint: Use the proof of the second boxed theorem in Part B as a guide in constructing the proof.]

G. Use a table of common logarithms and simplify:

1. $(86.7)^3$	2. $\sqrt[6]{265}$
3. $(0.362)^4$	4. $\sqrt[3]{0.567}$
5. $(23)^{-3}$	6. $(0.0259)^{-2}$
7. $\sqrt[7]{-362}$	8. $\sqrt[3]{(39.5)^5}$

Sample. $\dfrac{(27^3)(\sqrt{0.362})}{(0.0259)}$

Solution. $$\log \frac{(27^3)(\sqrt{0.362})}{(0.0259)} = \log(27^3)(\sqrt{0.362}) - \log(0.0259)$$

$$\log 27^3 + \log\sqrt{0.362} - \log 0.0259$$

$$= 3\log 27 + \frac{1}{2}\log 0.362 - \log 0.0259$$

(continued on next page)

[2-98]

G. (cont.)

9.	85.92	10.	18.90
11.	0.2458	12.	1.620×10^8

H.

1.	2.444	2.	-1.2970
3.	1.5729	4.	$\frac{3}{2}$

$$
\begin{aligned}
3 \log 27 &\overset{a}{=} 4.2942 \\
(+)\ \tfrac{1}{2} \log 0.362 &\overset{a}{=} 0.7794 - 1 \\
\log (27^3)(\sqrt{0.362}) &\overset{a}{=} 5.0736 - 1 = 6.0736 - 2 \\
\log 0.0259 &\overset{a}{=} 0.4132 - 2 = 0.4132 - 2 \\
\log \frac{(27^3)(\sqrt{0.362})}{(0.0259)} &\overset{a}{=} 5.6604 \\
\frac{(27^3)(\sqrt{0.362})}{(0.0259)} &\overset{a}{=} 4.575 \times 10^5 \\
&\doteq 457{,}500
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\log 27 &\doteq 1.4314 \\
& \qquad\quad 3 \\
3 \log 27 &\overset{a}{=} 4.2942 \\
\log 0.362 &\overset{a}{=} 1.5587 - 2 \\
& \qquad\quad \tfrac{1}{2} \\
\tfrac{1}{2} \log 0.362 &\overset{a}{=} 0.7794 - 1
\end{aligned}
}
$$

9. $\sqrt{(27.9)(265)}$

10. $\dfrac{(0.156)(3.62)^3}{\sqrt[5]{918}}$

11. $\dfrac{\left[(46.5)(\sqrt[3]{3.249})\right]^2}{(734)(26.3)}$

12. $\dfrac{(-265)^3(27.9)}{\sqrt[3]{-33}}$

H. Solve each equation.

Sample. $7^u = 116$

Solution.

$$
\begin{aligned}
7^u &= 116 \\
\log 7^u &= \log 116 \\
u \log 7 &= \log 116 \\
u &= \frac{\log 116}{\log 7} \\
&\overset{a}{=} \frac{2.0645}{0.8451} \\
&\overset{a}{=} 2.44
\end{aligned}
$$

An approximation to the root is 2.44.

1. $3^x = 15$

2. $4^{x+2} = 265$

3. $15^{3y-4} = 7$

4. $\log_9 27 = t$

EXPLORATION EXERCISES

Since $3^4 = 81$, $\left(3^2\right)^{\frac{4}{2}} = 81$ [Why?]. Another way of saying the foregoing is: Since $\log_3 81 = 4$, $\log_{3^2} 81 = \frac{4}{2}$ (that is, $\log_9 81 = 2$). Similarly,

$\log_{\sqrt{3}} 81 = 4 \div \frac{1}{2}$, and $\log_{3^\pi} 81 = \frac{4}{\pi}$ $[(3^\pi)^{\frac{4}{\pi}} = 3^4]$.

Use a table of common logarithms to verify each of the following.

1. $\log_{100} 2 \overset{a}{=} 0.1505$
2. $\log_{\sqrt{10}} 2 \overset{a}{=} 0.6020$
3. $\log_{10^{\sqrt{2}}} 34 = 1.0831$
4. $\log_{1000} 243 = 0.7952$
5. $\log_{0.1} 378 = -2.5775$
6. $\log_4 7.5 = 1.2873$

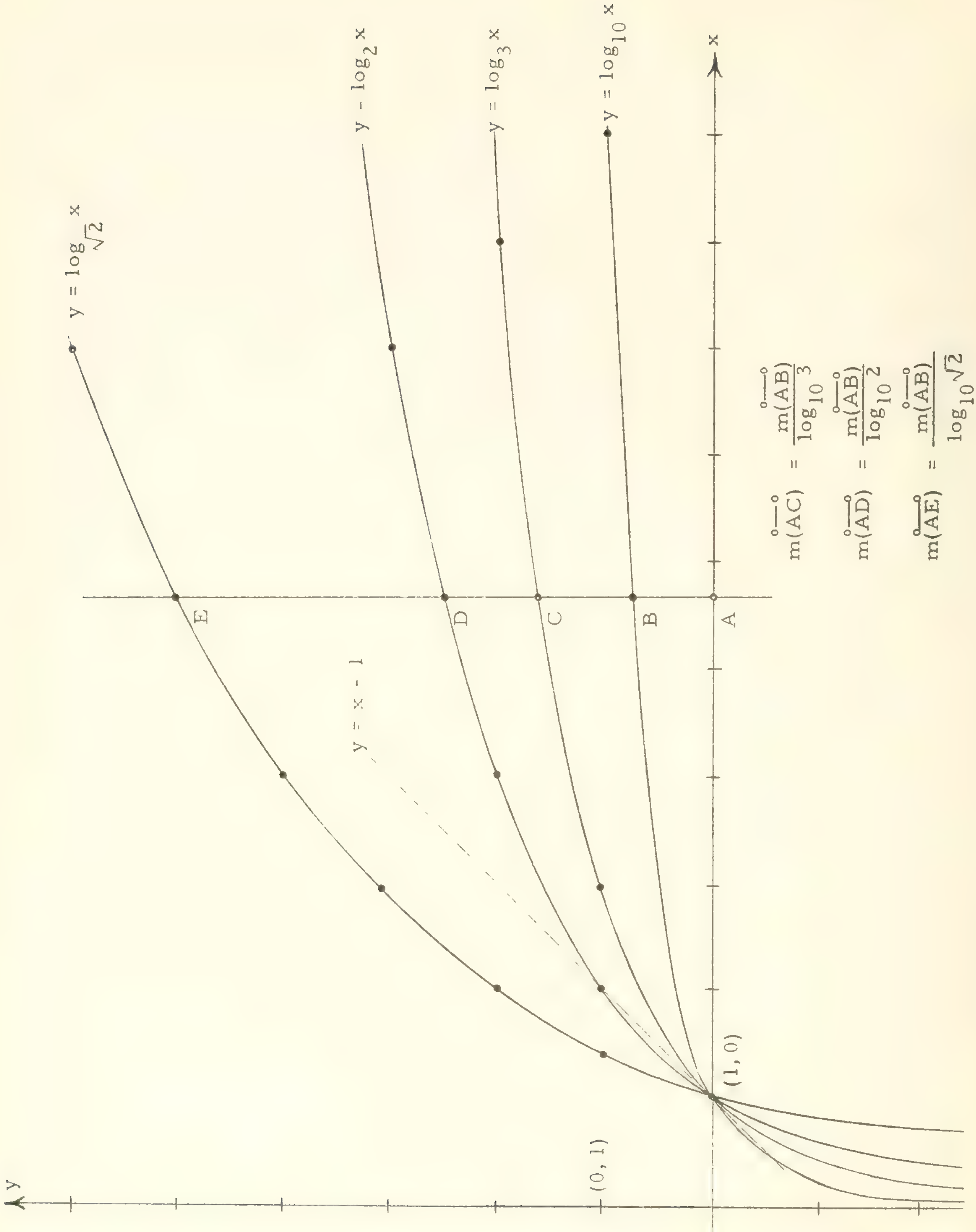
[Hint for Exercise 6: What is the exponent of that power of 10 which equals 4?]

2.09 Comparing logarithms to two bases.--The preceding Exploration Exercises suggest that if you know the common logarithm of each positive number then you can find the logarithm to any base of any positive number. We shall prove the following more general result.

> For every a such that $0 < a \neq 1$,
> for every b such that $0 < b \neq 1$,
> and for every $x > 0$,
>
> $$\log_b x = \frac{\log_a x}{\log_a b}.$$

Proof: By the defining principle,

$$a^{\log_a x} = x, \text{ and } a^{\log_a b} = b.$$



$y = \log_{\sqrt{2}} x$
$y = \log_2 x$
$y = \log_3 x$
$y = \log_{10} x$
$y = x - 1$
x
y
E
D
C
B
A
(1, 0)
(0, 1)
$m(\overset{\circ\!\!-\!\!\circ}{AC}) = \frac{m(\overset{\circ\!\!-\!\!\circ}{AB})}{\log_{10} 3}$
$m(\overset{\circ\!\!-\!\!\circ}{AD}) = \frac{m(\overset{\circ\!\!-\!\!\circ}{AB})}{\log_{10} 2}$
$m(\overset{\circ\!\!-\!\!\circ}{AE}) = \frac{m(\overset{\circ\!\!-\!\!\circ}{AB})}{\log_{10} \sqrt{2}}$

By the multiplication rule for exponents, since $\log_a b \neq 0$ [Why?],

$$a^{\log_a x} = \left(a^{\log_a b}\right)^{\frac{\log_a x}{\log_a b}}.$$

Consequently,

$$x = b^{\frac{\log_a x}{\log_a b}},$$

and, by the defining principle,

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

[Another proof of the same theorem goes as follows: $b^{\log_b x} = a^{\log_a x}$, so $\log_a\left(b^{\log_b x}\right) = \log_a\left(a^{\log_a x}\right)$. Consequently, $\log_b x \cdot \log_a b = \log_a x \cdot \log_a a$. Hence ....

The first proof shows how the theorem is related to the rules for exponents. The second proof shows that the theorem is a consequence of those rules which we have already established for logarithms. One proof of a theorem is sufficient to establish the theorem, but knowing alternative proofs may improve your understanding of a theorem.]

The theorem just proved has the following corollary.

> For every a and b such that
> $0 < a \neq 1$ and $0 < b \neq 1$,
> $\log_b a = 1/\log_a b$.

EXERCISES

A. 1. Show how the corollary above follows from the preceding theorem.

(continued on next page)

2. Use a table of common logarithms to find a decimal approximation.

(a) $\log_3 5$ (b) $\log_{13} 2$

(c) $\log_7 265$ (d) $\log_{0.5} 13$

(e) $\log_{98} 36$ (f) $\log_{\frac{1}{3}} 64$

B. Several logarithm curves are shown on page 2-100.

1. Why do all the curves pass through the point (1, 0)?
2. Fill in the blank so as to make the following sentence true.

   For each positive abscissa, the ordinate to the $\log_{\sqrt{2}}$ - curve is ________ the ordinate to the $\log_2$ - curve.

3. Why is it the case, for each of the log curves drawn on page 2-100, that points on the curves which are to the right of (1, 0) are above the x-axis?
4. State a sentence like that in Exercise 2 but with '$\log_{0.5}$' in place of '$\log_{\sqrt{2}}$'.
5. Sketch, on page 2-100, the $\log_{0.5}$ - curve.

C. The figure on page 2-100 illustrates the fact that all log curves have much the same shape. One can distinguish between two log curves by comparing how steep they are at (1, 0). Fill the blank in such a way as to make the following sentence true.

For every two numbers a and b, both greater than 1, the $\log_a$ - curve is steeper at (1, 0) than the $\log_b$ - curve just if __________ .

* * *

On the figure on page 2-100 there is shown (dotted) a portion of the line, $[\overleftrightarrow{(1, 0)(2, 1)}]$ which bisects one of the angles formed by the line whose equations are '$x = 1$' and '$y = 0$'. Since both (1, 0) and (2, 1)

belong to the $\log_2$ - curve, you can see that the line $\overleftrightarrow{(1, 0)(2, 1)}$ is <u>less steep</u> at (1, 0) than is the $\log_2$ - curve and it is <u>more steep</u> at (1, 0) than is the $\log_3$ - curve. You may guess that there is a number b (between 2 and 3) such that the line $\overleftrightarrow{(1, 0)(2, 1)}$ and the $\log_b$ - curve are equally steep at (1,0). This is correct, and the number b which has this property is always denoted by 'e'. We have seen that 2 < e < 3. Like the number π, e is not a rational number, but can be approximated as closely as we wish by rational numbers. For example,

$$e \stackrel{a}{=} 2.71828$$

Logarithms to the base e are called <u>natural logarithms</u>, and it is customary to abbreviate '$\log_e$' by 'ln'. When you study calculus you will see that the fact that the ln-curve and the bisector $\overleftrightarrow{(1,0)(2, 1)}$ are equally steep at (1,0) is a sufficient reason for calling e the "<u>natural</u>" <u>base for a system of logarithms</u>.

* * *

<u>D</u>. Use the fact that $e \stackrel{a}{=} 2.718$ and find rational approximations.

(a) ln 2 (b) ln 10 (c) ln 20 (d) ln 1.35

(e) ln (e^2) (f) ln 0.95 (g) ln 1 (h) ln (-2.4)

* * *

An equation of the line $\overleftrightarrow{(1,0)(2, 1)}$ is '$y = x - 1$'. You can see that as you go from left to right along any log curve (and, in particular, the ln-curve) the curve becomes less steep. It follows from this that, for every $x > 0$,

$$(*) \qquad \ln x \leq x - 1.$$

[In fact, for every x such that $0 < x \neq 1$, $\ln x < x - 1$.] Since $-\ln x = \ln\frac{1}{x}$ and, from (*), $\ln\frac{1}{x} \leq \frac{1}{x} - 1$, it follows that $-\ln x \leq \frac{1}{x} - 1$. Hence, for every $x > 0$, $\ln x \geq 1 - \frac{1}{x}$. We can combine these two inequalities for '$\ln x$' and say that,

for every $x > 0$,

$$1 - \frac{1}{x} \leq \ln x \leq x - 1,$$

that is,

$$\frac{x - 1}{x} \leq \ln x \leq x - 1.$$

Now replacing, for convenience, 'x' by '1 + x', we see that

> For every $x > -1$,
>
> $$\frac{x}{1+x} \leq \ln(1+x) \leq x.$$

Another way of saying part of what the boxed statement says is:

For every $x > -1$ and $\neq 0$,

$\frac{1}{x} \ln(1+x)$ is between $\frac{1}{x}\left(\frac{x}{1+x}\right)$ and $\frac{1}{x} \cdot x$,

that is, it is between $\frac{1}{1+x}$ and 1.

Since, for every x and y, if $x \leq y$ then $e^x \leq e^y$,

for every $x > -1$ and $\neq 0$,

$e^{\frac{1}{x}\ln(1+x)}$ is between $e^{\frac{1}{1+x}}$ and e^1.

Since, for every x; $e^{\frac{1}{x}\ln(1+x)} = \left[e^{\ln(1+x)}\right]^{\frac{1}{x}} = (1+x)^{\frac{1}{x}}$,

for every $x > -1$ and $\neq 0$,

$(1+x)^{\frac{1}{x}}$ is between $e^{\frac{1}{1+x}}$ and e^1.

Now, it is clear that, for x sufficiently close to 0, $\frac{1}{1+x}$ will differ from 1 by as little as we please. Hence, by characteristic (ii) of a smooth curve, if x is sufficiently close to 0 then $e^{\frac{1}{1+x}}$ will differ from e^1 by as little as we please. Hence, the same is true of $(1+x)^{\frac{1}{x}}$. For example,

x	$(1+x)^{\frac{1}{x}}$
0.50	2.250
0.10	2.594
0.01	2.705
0.001	2.717
-----	$e \stackrel{a}{=} 2.718$
-0.001	2.720
-0.01	2.732
-0.10	2.868
-0.50	4.000

Mathematicians abbreviate this statement (that, for x sufficiently close to 0, $(1+x)^{\frac{1}{x}}$ differs as little as we please from e) by writing:

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e.$$

REVIEW EXERCISES

A. Geometric progressions.

You have learned [pages 1-26, and 1-21, 1-22] that a progression is a sequence of numbers which is ordered by some relation in the same way that < orders the counting numbers. On page 1-37 you learned about progressions of a special kind called 'arithmetic progressions'. An arithmetic progression is a progression for which the difference between successive terms is constant. Now you will study another kind of progression.

(1) $2, 4, 8, 16, 32, 64, \ldots$

(2) $1, -3, 9, -27, 81, -243, \ldots$

(3) $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

(4) $1, -1, 1, -1, 1, -1, \ldots$

(5) $\pi, \pi, \pi, \pi, \pi, \pi, \ldots$

Progressions such as these are called geometric progressions. Notice that a characteristic property of geometric progressions is this:

> The quotient obtained by dividing any term of a geometric progression into its follower is the same as the quotient obtained by dividing any other term into its follower.

This is usually abbreviated to:

> The quotient of successive terms of a G. P. is constant. This quotient is the common ratio of the G. P.

['ratio' is often used in place of 'quotient'; for every number a and every $b \neq 0$, the ratio of a to b is the quotient obtained by dividing a by b.] For example, for the G. P. (1) the common ratio is 2, and for the G. P. (2) the common ratio is -3. What are the common ratios for the G. P.'s (3), (4) and (5)?

For all real numbers a and r, the successive terms of the G. P. whose first term is a and whose common ratio is r are the values of:

$$ar^{x-1},$$

for the values 1, 2, 3, etc. of 'x'.

1. Fill in the blanks in each of the following so that the result gives a geometric progression.

   (a) 1, 2, ___, 8, ___, ___, ...

   (b) 3, ___, ___, 81, ___, ___, ...

   (c) -2, ___, $-\frac{1}{2}$, ___, ___, ___, ... [(c) has two solutions.]

   (d) 3, $3\sqrt{2}$, ___, ___, ___, ___, ...

   (e) $\sqrt{3}$, ___, ___, 9, ___, ___, ...

   (f) $\sqrt{3}$, ___, ___, -9, ___, ___, ...

2. In filling the blanks between '3' and '81' in part (b) of Exercise 1 you "inserted two geometric means between 3 and 81". In part (c) you inserted one geometric mean between -2 and $-\frac{1}{2}$ (and found that there were two ways in which you could do this).

   (a) Insert two geometric means between 2 and 4.

   (b) Insert three geometric means between 1 and 8.

   (c) Insert four geometric means between -1 and 32.

   (d) Insert three geometric means between -1 and 8.

3. When can you insert three geometric means between two numbers? Any odd number of geometric means?

4. If three numbers, all positive or all negative, are consecutive terms of a G. P. then the second is called the <u>geometric mean of the first and third</u>.

   (a) Find the geometric mean of 2 and 4.

   (b) Find the geometric mean of -1 and -9.

   (c) Prove the following generalization:

> For every $x > 0$ and every $y > 0$, the geometric mean of x and y is
>
> $$\sqrt{xy}.$$

5. Suppose that the passing grade in your class is 63 and that you have taken two examinations on one of which your grade was 40, on the other 90. When "averaging" these grades, would you rather that your teacher used their arithmetic mean (page 1-39) or their geometric mean?

6. Prove that the arithmetic mean of two positive numbers is always greater than their geometric mean, <u>i.e.</u>, prove:

> For every x and y such that $x > 0$, $y > 0$, and $x \neq y$,
>
> $$\frac{x+y}{2} > \sqrt{xy}.$$

[Hint: If $x > 0$ and $y > 0$ then $x + y - 2\sqrt{xy} = (\sqrt{x} - \sqrt{y})^2$ (Why?).]

7. Since the successive terms of the G. P. whose first term is a and whose common ratio is r are the values of 'ar^{x-1}' for the values 1, 2, 3, etc. of 'x',

> For every counting number n, the sum of the first n terms of the G. P. whose first term is a and whose common ratio is r is
>
> $$\sum_{x=1}^{n} ar^{x-1}.$$

For example, $3 + 3\sqrt{2} + 6 + 6\sqrt{2} = \sum_{x=1}^{4} 3(\sqrt{2})^{x-1}$.

We sometimes need a simple way to compute the sum of a given number of consecutive terms of a G. P. Since $\sum_{x=1}^{n} ar^{x-1} = a\sum_{x=1}^{n} r^{x-1}$, what we need is a simple formula for '$\sum_{x=1}^{n} r^{x-1}$'. Now, for every r, and for every counting number n,

$$(r-1)\sum_{x=1}^{n} r^{x-1} = \sum_{x=1}^{n} r^{x} - \sum_{x=1}^{n} r^{x-1} \qquad \text{[Why?]}$$

But

$$\sum_{x=1}^{n} r^{x} = \sum_{y=2}^{n+1} r^{y-1} \qquad \text{[Why?]}$$

$$= \left(\sum_{y=2}^{n} r^{y-1}\right) + r^{n} \qquad \text{[Why?]}$$

$$= \left[\left(\sum_{y=1}^{n} r^{y-1}\right) - r^{0}\right] + r^{n} \qquad \text{[Why?]}$$

$$= \sum_{y=1}^{n} r^{y-1} + (r^{n} - 1) \qquad \text{[Why?].}$$

Hence,

$$(r-1)\sum_{x=1}^{n} r^{x-1} = \left[\sum_{y=1}^{n} r^{y-1} + (r^{n} - 1)\right] - \sum_{x=1}^{n} r^{x-1}$$

$$= r^{n} - 1.$$

Consequently:

> For every $r \neq 1$, and every counting number n,
>
> $$\sum_{x=1}^{n} r^{x-1} = \frac{r^n - 1}{r - 1}.$$

> For every a, every $r \neq 1$, and every counting number n, the sum of the first n terms of the G. P. whose first term is a and whose common ratio is r is
>
> $$\frac{a(r^n - 1)}{r - 1}, \left(\text{or: } \frac{a(1 - r^n)}{1 - r}\right).$$

(a) Use the second boxed theorem to compute

(i) $2 + 4 + 8 + 16 + 32 + 64$

(ii) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64}$

(iii) $1 - 1 + 1 - 1 + 1 - 1 + 1$

(b) What is the sum of the first n terms of the G. P. whose first term is a and whose common ratio is 1?

(c) Use mathematical induction to prove the first of the boxed theorems above.

8. (a) Find the 9th term in the G. P. $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \ldots$.

(b) If the 8th term in a G. P. is $\frac{1}{2}$ and if the common ratio is $\frac{1}{2}$, find the first term.

(continued on next page)

(c) Find the 20th term of the G. P. $3\sqrt{3}$, 9, $9\sqrt{3}$, ...

(d) If the third term of a G. P. is 2^{-3} and the sixth term is 2^{-9}, find the first term.

(e) Find the sum of the first 10 terms of the G. P. 1000, 100, 10, ...

(f) What is the sum of the first 15 terms of the G. P. whose second term is $\frac{1}{2}$ and whose fifth term is $\frac{1}{16}$?

(g) Find a formula for the sum of the first 10 terms of the G. P. t, s^2t^2, s^4t^3, ...

(h) Which term of the G. P. $-5^{\frac{1}{5}}$, $5^{\frac{12}{5}}$, $-5^{\frac{23}{5}}$, ... is 5^{20}?

(i) True or False?

(1) If each term of an arithmetic progression is multiplied by a real number, the resulting progression is an A. P.

(2) If each term of a geometric progression is multiplied by a real number, the resulting progression is a G. P.

(j) Are there progressions which are both arithmetic progressions and geometric progressions?

B. Factoring.

You know that, for every a and b, $a^2 - b^2 = (a - b)(a + b)$.

1. Prove that, for every a and b, $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, and $a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3)$.

2. Use the first boxed theorem of Exercise 7 of Part A to factor each of the following expressions.

Sample. $r^7 - 1$

Solution. For every $r \neq 1$,

$$\frac{r^7 - 1}{r - 1} = \sum_{x=1}^{7} r^{x-1}$$

$$= 1 + r + r^2 + r^3 + r^4 + r^5 + r^6.$$

(continued on next page)

Thus, for every $r \neq 1$,

$$r^7 - 1 = (r - 1)(1 + r + r^2 + r^3 + r^4 + r^5 + r^6).$$

Since the foregoing equation is satisfied by each real number, including 1, the factored form of '$r^7 - 1$' is '$(r - 1)(1 + r + r^2 + r^3 + r^4 + r^5 + r^6)$'.

(a) $r^3 - 1$ (b) $x^5 - 1$ (c) $1 - y^8$

(d) $a^5 - b^5$ [Hint: For every a and $b \neq 0$, $a^5 - b^5 = b^5\left[\left(\frac{a}{b}\right)^5 - 1\right]$.]

(e) $x^7 - y^7$

3. (a) Prove that, for every r and every counting number n,

$$r^n - 1 = (r - 1)\sum_{x=0}^{n-1} r^x.$$

(b) Prove that, for every a and b, and every counting number n,

$$a^n - b^n = (a - b)\sum_{x=0}^{n-1} a^x \cdot b^{n-1-x}.$$

4. One consequence of the result of Exercise 3(b) is that if 'n' in '$a^n - b^n$' is replaced by a name for any counting number then the resulting expression has '$a - b$' as a factor. [In order to explain the notion of a factor of an expression, we need the notion of a polynomial in 'a' and 'b' with integral coefficients. Such polynomials are expressions of a certain kind. The following are examples:

$$-2,\ a,\ b^2,\ a^2b,\ 3a^2b^4,\ 2 - a,\ 3ab + b^2 - 2ab^5,\ a^4 - b^4.$$

A polynomial in 'a' and 'b' with integral coefficients is an expression which can be constructed in the following way: We start with the letters 'a' and 'b', and numerals for integers, and with exponentials each of which has either 'a' or 'b' as base symbol and a numeral for a positive integer as exponent symbol. We then form "products" by connecting two or more of these by times signs (as usual, we

most often use as a times sign a very narrow blank space). A polynomial in 'a' and 'b' with integral coefficients is either one of the expressions already described, or is an expression obtained by connecting two or more of them with plus or minus signs.

You should now check to see that each of the expressions given as examples above is a polynomial in 'a' and 'b' with integral coefficients.

Other kinds of polynomials can be defined in a manner similar to the above. (An essential point is that, for any polynomial, the exponent symbols are always numerals for positive integers.) Give some examples of polynomials in 'x' with real coefficients, and then formulate an appropriate definition.

(From here to the end of the bracket, 'polynomial' is short for 'polynomial with integral coefficients'.)

We can now explain the notion of a <u>factor of a polynomial</u>. First, an example:

> '$a - b$' and '$a^2 + ab + b^2$' are factors of the polynomial '$a^3 - b^3$' because the first two expressions are polynomials and, for every a and b,
>
> $$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

In general, if we use 'P' and 'F' as abbreviations for some two polynomials then the second of these polynomials is a factor of the first if there is a polynomial such that, using 'G' as an abbreviation for it,

$$P = F \cdot G$$

for all values of the variables which occur in the three polynomials.]

Factor each of the following expressions.

<u>Sample</u>. $x^4 - 16$

<u>Solution</u>. Since, for every a and b,

$$a^2 - b^2 = (a - b)(a + b), \text{ for every } x,$$

$$x^4 - 16 = (x^2 - 4)(x^2 + 4).$$

(continued on next page)

Hence, '$x^4 - 16$' has '$x^2 - 4$' and '$x^2 + 4$' as factors. However, '$x^2 - 4$' has, for the same reason, the factors '$x - 2$' and '$x + 2$', so it is easily seen that each of these is a factor of '$x^4 - 16$'. [For instance, for every x,

$$x^4 - 16 = (x - 2)[(x + 2)(x^2 + 4)]$$

$$= (x - 2)(x^3 + 2x^2 + 4x + 8)$$

and '$x^3 + 2x^2 + 4x + 8$' is a polynomial in 'x' with integral coefficients. Hence '$x - 2$' is a factor of '$x^4 - 16$'.] On the other hand, it can be proved that '$x^2 + 4$' has no factors except itself, '1', and '-1'.

The most convenient form of answer for this Sample is:

$$x^4 - 16 = (x - 2)(x + 2)(x^2 + 4).$$

(a) $x^4 - 25$ (b) $1 - y^2$

(c) $x^3 - 8$ (d) $a^3 + b^3$

(e) $x^2 + 2xy + y^2$ (f) $x^2 - 5x + 6$

(g) $t^3 + t^2w - tw^2 - w^3$ (h) $2a^2 - 5ab - 3b^2$

[Hint for (g): $t^3 + t^2w - tw^2 - w^3 = t^2(t + w) - (t + w)w^2$.]

5. Most simple factoring problems can be solved by applying the following theorems:

I. For every a, b, and c, $ab + ac = a(b + c)$.

II. For every a and b, $a^2 - b^2 = (a - b)(a + b)$, and
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

III. For every a and b, $a^2 + 2ab + b^2 = (a + b)^2$.

IV. For every a, b, and c, $x^2 + (b + c)x + bc = (x + b)(x + c)$.

You probably used II in solving parts (a), (b), (c), and (d) of Exercise 4, III in solving part (e), and IV in solving part (f). In solving part (g) you might have used I, and then II.

You may have solved part (h) by guess work based in part on the methods used to solve parts (e) and (f). One attack is to guess that if '$2a^2 - 5ab - 3b^2$' has any factors then it has two, one of which begins '2a ---' and the other 'a ---'. In fact, the first must be one of the four polynomials:

$$2a + b, \quad 2a - b, \quad 2a + 3b, \quad 2a - 3b,$$

and the second must be one of the polynomials:

$$a + b, \quad a - b, \quad a + 3b, \quad a - 3b.$$

A little experimentation shows that the proper choices are '$2a + b$' and '$a - 3b$'. Theorem III, on the previous page, is a special case of IV and this, in turn, can be generalized to obtain a theorem which will handle exercises like (h).

V. For every a, b, c, d, x, and y,

$$acx^2 + (ad + bc)xy + bdy^2 = (ax + by)(cx + dy).$$

Factor each of the following.

(a) $64a^2 + 80ab + 25b^2$ (b) $16 - y^4$

(c) $7 - 7x^3$ (d) $21x^2 + xy - 10y^2$

(e) $20s^2 - 41st + 20t^2$ (f) $8k^3 - 27$

(g) $36 - x^2 - 2xy - y^2$ (h) $10y^4 + 29y^2x^2 + 10x^4$

(i) $\left(x^2\right)^3 - \left(y^2\right)^3$ (j) $\left(x^3\right)^2 - \left(y^3\right)^2$

(k) $x^2 + 2xy + y^2 - a^2 - 2ab - b^2$

(l) $(x + 1)^3 - (y + 1)^3$

C. Bernoulli's Inequality.

As an application of mathematical induction we shall prove that

for every integer $n > 1$, and every h such that $-1 \leq h \neq 0$,

$$(1 + h)^n > 1 + nh.$$

[This is called 'Bernoulli's Inequality'. The theorem above has many important consequences, among them that of Exercise 4 which we need for Part D.]

Proof: We are concerned with the property expressed by:

for every h such that $-1 \leq h \neq 0$,

$$(1 + h)^{\cdots} > 1 + \ldots h,$$

and wish to show that every integer ≥ 2 has this property.

(i) 2 has the property.

For $(1 + h)^2 = 1 + 2h + h^2 > 1 + 2h$, if $h \neq 0$.

(ii) Suppose, for some integer k, it is the case that

for every h such that $-1 \leq h \neq 0$,

$$(1 + h)^k > 1 + kh.$$

Then, for this k, it is the case that, for every h such that $-1 \leq h \neq 0$,

$$\begin{aligned}
(1 + h)^{k+1} &= (1 + h)^k(1 + h) \\
&\geq (1 + kh)(1 + h) \quad [(1 + h)^k > 1 + kh \text{ and } 1 + h \geq 0] \\
&= 1 + (k + 1)h + kh^2 \\
&> 1 + (k + 1)h, \text{ if } k > 0 \quad [kh^2 > 0 \text{ if } k > 0, \text{ since } h \neq 0].
\end{aligned}$$

Hence, the property is hereditary over the set of positive integers. Since it holds for 2 it follows by mathematical induction that it holds for every integer $n \geq 2$.

1. Prove that, for every integer $n \geq 0$, and every number $h \geq -1$,

$$(1 + h)^n \geq 1 + nh.$$

2. Prove that, for every integer $n \geq 0$, and every number $h \leq 1$,

$$(1 - h)^n \geq 1 - nh.$$

3. Prove that, for every $b > 1$, every y, and every integer $n \geq 0$,

$$b^n > y \text{ if } n > \frac{y - 1}{b - 1}.$$

[Hint: By Exercise 1, $b^n \geq 1 + n(b - 1)$ if $b - 1 \geq -1$ and $n \geq 0$.]

4. Prove that, for every r such that $0 < r < 1$, every $t > 0$, and every integer $n \geq 0$,

$$0 < r^n < t \text{ if } n > \frac{r(1 - t)}{t(1 - r)}.$$

[Hint: In Exercise 3, substitute '$\frac{1}{r}$' for 'b' and '$\frac{1}{t}$' for 'y'.]

[What conclusion can you draw if $r = 0$?]

D. Geometric Progressions and Repeating Decimals.

You have learned that the repeating decimal:

$$0.333\ldots3\ldots \quad (\text{or: } 0.\dot{3})$$

is a name for the number $\frac{1}{3}$. Why should this be so? To answer this question, we begin by noting that, for example, the "finite" decimal '0.568' is a name for

$$\frac{5}{10} + \frac{6}{100} + \frac{8}{1000} \quad .$$

So, perhaps the "infinite" decimal '$0.\dot{3}$' is a name for

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots,$$

where '...' means that we are supposed to keep on adding one number after another, each number being one-tenth of the preceding. But, if this is so, '$0.\dot{3}$' is a name for the sum of infinitely many numbers, $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$, etc., and what can it mean to add infinitely many numbers? Until now you have only added numbers two at a time (remember, for

example, that '2 + 3 + 7' means '(2 + 3) + 7', so in order to simplify '2 + 3 + 7' you first note that 2 + 3 = 5 and then that 5 + 7 = 12). Perhaps, the best we can do is to add up a few of the numbers $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$, etc. and see what happens.

$$\frac{3}{10} = \frac{3}{10} \qquad\qquad \frac{1}{3} - \frac{3}{10} = \frac{1}{30}$$

$$\frac{3}{10} + \frac{3}{100} = \frac{33}{100} \qquad\qquad \frac{1}{3} - \frac{33}{100} = \frac{1}{300}$$

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} = \frac{333}{1000} \qquad\qquad \frac{1}{3} - \frac{333}{1000} = \frac{1}{3000}$$

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} = \frac{3333}{10000} \qquad\qquad \frac{1}{3} - \frac{3333}{10000} = \frac{1}{30000}$$

If you compare the first number, the sum of the first two, the sum of the first three, etc., each with $\frac{1}{3}$, you see that each of these sums is a closer approximation to $\frac{1}{3}$ than the preceding sum is, and that, apparently, you can get as close an approximation to $\frac{1}{3}$ as you wish by adding sufficiently many of the numbers $\frac{3}{10}$, $\frac{3}{100}$, etc.

In fact, the numbers $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$, ... form a G. P. whose first term is $\frac{3}{10}$ and whose common ratio is $\frac{1}{10}$. Consequently, for any counting number n, the sum of the first n terms, $\sum_{x=1}^{n} \frac{3}{10^x}$, is

$$\frac{\frac{3}{10}\left[1 - \left(\frac{1}{10}\right)^n\right]}{1 - \frac{1}{10}} \qquad \text{[Why?].}$$

That is, $$\frac{1}{3} - \frac{1}{3 \times 10^n} \qquad \text{[Why?].}$$

You now know that, for every counting number n,

$$(*) \qquad \frac{1}{3} - \sum_{x=1}^{n} \frac{3}{10^x} = \frac{1}{3 \times 10^n}.$$

From Exercise 4 of Part C you know that, for every $t > 0$ and every $n \geq 0$,

$$\frac{1}{10^n} < 3t \quad \text{if} \quad n > \frac{\frac{1}{10}(1 - 3t)}{3t\left(1 - \frac{1}{10}\right)} = \frac{1 - 3t}{27t}.$$

Consequently, for every $t > 0$, $\frac{1}{3 \times 10^n} < t$ for these same values of 'n', and, by equation (*),

$$\frac{1}{3} - \sum_{x=1}^{n} \frac{3}{10^x} < t$$

if n is sufficiently large. Since, for $t > 0$,

$$-t < 0 < \frac{1}{3} - \sum_{x=1}^{n} \frac{3}{10^x},$$

we can say that, for every $t > 0$ there exists an integer N such that

$$\left| \frac{1}{3} - \sum_{x=1}^{n} \frac{3}{10^x} \right| < t \quad \text{if} \quad n > N.$$

Mathematicians abbreviate this by:

$$\lim_{x \to \infty} \left(\sum_{x=1}^{n} \frac{3}{10^x} \right) = \frac{1}{3}$$

and also by:

$$\sum_{x=1}^{\infty} \frac{3}{10^x} = \frac{1}{3}.$$

Getting back now to the repeating decimal, '$0.\dot{3}$', this is just a further abbreviation for '$\sum_{x=1}^{\infty} \frac{3}{10^x}$' which is a name for the number which is approximated arbitrarily closely by the successive values of '$\sum_{x=1}^{n} \frac{3}{10^x}$'. Since this number is $\frac{1}{3}$, '$0.\dot{3}$' is one of the many names of $\frac{1}{3}$.

We have discussed above the G. P. whose first term is .3 and whose common ratio is $\frac{1}{10}$, and have seen how to make sense of 'the sum of all the terms' of this G. P. Let us now consider a more general situation. For every $a \neq 0$ and $r \neq 0$, there is a G. P. whose first term is a and whose common ratio is r. For every counting number x, the <u>xth</u> term of this G. P. is

$$ar^{x-1},$$

and for every counting number n, the sum of its first n terms, $\sum_{x=1}^{n} ar^{x-1}$, is

$$\frac{a(1 - r^n)}{1 - r},$$

that is,

$$\frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Hence,

$$\frac{a}{1 - r} - \sum_{x=1}^{n} ar^{x-1} = \frac{ar^n}{1 - r}$$

$$= \frac{r^n}{b}, \text{ say, where } b = \frac{1 - r}{a}.$$

Now, <u>if</u> <u>$0 < r < 1$</u>, you know from Exercise 4 of Part C that, for every $t > 0$, and every $n \geq 0$,

$$r^n < |b|t \quad \text{if} \quad n > \frac{r(1 - |b|t)}{|b|t(1 - r)}.$$

Consequently, for every $t > 0$,

$$\left|\frac{a}{1 - r} - \sum_{x=1}^{n} ar^{x-1}\right| = \frac{|a|r^n}{1 - r}$$

$$= \frac{r^n}{|b|}$$

$$< t \quad \text{if} \quad n \text{ is sufficiently large.}$$

As before (in the case $a = .3$, $r = \frac{1}{10}$), we can abbreviate this statement to:

$$\lim_{n \to \infty} \sum_{x=1}^{n} ar^{x-1} = \frac{a}{1 - r}$$

or, further, to:

$$\sum_{x=1}^{\infty} ar^{x-1} = \frac{a}{1-r}.$$

It is not very difficult, now, to carry through the argument of the preceding paragraph under the less restrictive hypothesis '$|r| < 1$', instead of '$0 < r < 1$'. Consequently,

> For every a and every r such that $|r| < 1$,
>
> $$\sum_{x=1}^{\infty} ar^{x-1} = \frac{a}{1-r},$$
>
> i.e., for every $t > 0$ there exists an N such that
>
> $$\left|\frac{a}{1-r} - \sum_{x=1}^{n} ar^{x-1}\right| < t$$
>
> if $n > N$.

Find a simpler name for the number named by each of the following repeating decimals.

Sample 1. $0.\dot{2}\dot{3}\dot{6}$ (i.e.: 0.236 236 236 ...)

Solution. This decimal is a name for the sum of all the terms of the G. P.

$$\frac{236}{1000}, \frac{236}{1000000}, \ldots$$

whose first term is $\frac{236}{1000}$ and whose common ratio is $\frac{1}{1000}$. Another name for this number is:

$$\sum_{x=1}^{\infty} \frac{236}{1000}\left(\frac{1}{1000}\right)^{x-1}$$

and, by the boxed theorem above, the number in question is

$$\frac{\frac{236}{1000}}{1 - \frac{1}{1000}}, \text{ or } \frac{236}{999}.$$

Sample 2. $1.2\dot{4}\dot{5}$

Solution.
$$1.2\dot{4}\dot{5} = 1.2 + 0.0\dot{4}\dot{5}$$
$$= \frac{12}{10} + \frac{\frac{45}{1000}}{1 - \frac{1}{100}}$$
$$= \frac{12}{10} + \frac{45}{990}$$
$$= \frac{6}{5} + \frac{1}{22}$$
$$= \frac{137}{110}.$$

[Check this answer by dividing 110 into 137.]

1. $0.\dot{6}$
2. $0.\dot{0}\dot{7}$
3. $92.\dot{8}$
4. $1.63\dot{4}\dot{8}$
5. $0.\dot{1}\dot{4}\dot{2}\dot{8}\dot{5}\dot{7}$
6. $9.003\dot{5}$

From your work in solving these exercises above you can see that

each repeating decimal is a name for some rational number.

From your experience with the process of "long division" you can see that

each rational number has a repeating decimal as one of its names.

E. Factorials.

There is a very useful progression, whose terms are called factorials, which can be defined recursively [see page 1-15] as follows.

$$\boxed{\begin{array}{c} 0! = 1, \\ \text{and, for every integer } k \geq 0, \\ (k+1)! = k! \cdot (k+1). \end{array}}$$

For example, $5! = (4 + 1)! = 4! \cdot 5$

$= (3 + 1)! \cdot 5$

['0!' is read 'zero factorial', $= 3! \cdot 4 \cdot 5$

'5!' is read 'five factorial', $= (2 + 1)! \cdot 4 \cdot 5$

'(4 + 1)!' is read 'four plus $= 2! \cdot 3 \cdot 4 \cdot 5$

one factorial', etc.] $= (1 + 1)! \cdot 3 \cdot 4 \cdot 5$

$= 1! \cdot 2 \cdot 3 \cdot 4 \cdot 5$

$= (0 + 1)! \cdot 2 \cdot 3 \cdot 4 \cdot 5$

$= 0! \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$

$= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$

1. Fill in the following table.

x	0	1	2	3	4	5	6	7	8
x!				6					

2. Simplify.

(a) $\frac{5!}{3!}$ (b) $\frac{12!}{11!}$ (c) $\frac{3!}{0!}$

(d) $\frac{9!}{13!}$ (e) $\frac{(8 - 2)!}{8!\ 6!}$ (f) $\frac{5!}{1!\ 2!\ 3!\ 4!\ 5!}$

(g) $\frac{(3 - 2)!}{(3 - 2)!}$ (h) $\frac{19!}{10!\ 9!}$ (i) $\frac{9!}{6!\ 3!}$

3. In Part F of these Exercises we shall find an important use for the following defining principle.

> For every integer $n \geq 0$, and every integer p such that $0 \leq p \leq n$,
>
> $$\binom{n}{p} = \frac{n!}{p!\,(n - p)!}.$$

'$\binom{5}{3}$' is read: the binomial coefficient five over three. [The reason for the phrase 'binomial coefficient' will appear in Part F.]

Simplify each of the following.

(a) $\binom{5}{3}$ (b) $\binom{6}{2}$ (c) $\binom{6}{4}$

(d) $\binom{9}{0}$ (e) $\binom{9}{1}$ (f) $\binom{9}{2}$

(g) $\binom{9}{3}$ (h) $\binom{9}{4}$ (i) $\binom{9}{5}$

(j) $\binom{9}{6}$ (k) $\binom{10}{6}$ (l) $\binom{12}{12}$

4. Fill the blanks in the following table so that the number $\binom{n}{p}$ is listed at the intersection of "row n" and "column p".

n \ p	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2					
3		3					
4							
5							
6			15				1

5. Prove that, for every integer $n \geq 0$,

$$\binom{n}{0} = 1 = \binom{n}{n}.$$

6. Prove that, for every integer $n \geq 0$ and every integer p such that $0 \leq p \leq n$,

$$\binom{n}{n-p} = \binom{n}{p}.$$

7. Prove that, for every integer $n \geq 1$, and every integer p such that $1 \leq p \leq n$,

$$\binom{n}{p-1} + \binom{n}{p} = \binom{n+1}{p}.$$

[An instance of this generalization is '$\binom{2}{0} + \binom{2}{1} = \binom{3}{1}$', and this is the case because $1 + 2 = 3$ (see row 2 and row 3 of the table above. Use the table to check other instances.]

[Hint: To solve Exercise 7, note that, by the defining principle,

$$\binom{n}{p-1} + \binom{n}{p} = \frac{n!}{(p-1)!\,(n-p+1)!} + \frac{n!}{p!\,(n-p)!}$$

and that, for example,

$$\frac{n!}{(p-1)!\,(n-p+1)!} = \frac{n!\cdot p}{p!\,(n-p+1)!}\ .]$$

F. The Binomial Theorem.

Each of the following equations holds for all values of 'a' and 'b'.

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

1. Check the last of the above formulas by simplifying '(a + b)(a + b)(a + b)(a + b)' in two ways.

2. Compare the polynomials on the right sides of the equations above with the table of Exercise 4, Part E.

(continued on next page)

3. Guess a polynomial which, when you write it on the right side of:

$$(a + b)^5 = \qquad ,$$

will yield an equation which holds for all values of 'a' and 'b'.

4. Prove that your guess is correct by simplifying:

$$(a + b)(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4).$$

5. Notice that the fourth of the equations which precede Exercise 1 could be rewritten:

$$(a + b)^3 = 1 \cdot a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$

or:

$$(a + b)^3 = \binom{3}{0}a^3b^0 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}a^0b^3$$

or:

$$(a + b)^3 = \binom{3}{0}a^{3-0}b^0 + \binom{3}{1}a^{3-1}b^1 + \binom{3}{2}a^{3-2}b^2 + \binom{3}{3}a^{3-3}b^3$$

or:

$$(a + b)^3 = \sum_{p=0}^{3}\binom{3}{p}a^{3-p}b^p.$$

Rewrite the other four equations preceding Exercise 1 in this last way, and do the same for the equation you wrote in answer to Exercise 3. [You may have more trouble with the equation '$(a + b)^0 = 1$' than with the others. If so, then rewrite some or all of the others first and then come back to this one.]

6. Exercise 6 suggests that the following is a theorem.

> For every a and b, and every integer $n \geq 0$,
>
> $$(a + b)^n = \sum_{p=0}^{n}\binom{n}{p}a^{n-p}b^p.$$

This is the case. The boxed statement is called 'the binomial theorem' ['(a + b)' is a binomial]. We shall prove the binomial theorem later. At present, you should get acquainted with it. In each of the following problems you are to write a statement equivalent to an instance of the binomial theorem, but not containing the symbol '$\sum$'.

Sample. $(x - y)^6$

Solution. $$(x - y)^6 = (x + (-y))^6 = \sum_{p=0}^{6} \binom{6}{p} x^{6-p}(-y)^p$$

$$= \binom{6}{0}x^6 + \binom{6}{1}x^5(-y) + \binom{6}{2}x^4(-y)^2$$

$$+ \binom{6}{3}x^3(-y)^3 + \binom{6}{4}x^2(-y)^4 + \binom{6}{5}x(-y)^5$$

$$+ \binom{6}{6}(-y)^6$$

$$= x^6 + 6x^5(-y) + \frac{6 \cdot 5}{1 \cdot 2}x^4(-y)^2 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}x^3(-y)^3$$

$$+ \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4}x^2(-y)^4 + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}x(-y)^4 + (-y)^6$$

$$= x^6 + 6x^5(-y) + 15x^4(-y)^2 + 20x^3(-y)^3$$

$$+ 15x^2(-y)^4 + 6x(-y)^5 + (-y)^6$$

$$= x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4$$

$$- 6xy^5 + y^6.$$

[Recall that, for every integer $n \geq 0$, $\binom{n}{0} = 1$. Notice, also, that, if $n \geq 1$, $\binom{n}{1} = n$. In general, if $n \geq p > 0$ then

(*) $$\binom{n}{p} = \frac{n(n-1) \ldots (n-p+1)}{1 \cdot 2 \ldots p}$$ [Why?].

Note that if numerals are substituted for 'n' and 'p' in the expression suggested by the right side of equation (*) then the resulting fraction has the same number of factors in its numerator and denominator. Finally, recall Exercise 6 of Part E, which tells you that, when you use the binomial theorem to "expand" a binomial exponential, you need calculate the values of only about half of the binomial coefficients which occur in the expansion [Why "<u>about</u> half"?].]

(a) $(a + b)^8$ (b) $(x - y)^7$ (c) $(2a + b)^4$

(d) $(3x - 2y)^5$ (e) $(\frac{1}{2}x + \frac{1}{3}y)^5$ (f) $(7a - 1)^5$

(g) $(x\sqrt{2} + y\sqrt{6})^6$ (h) $(3 - \sqrt{3})^4$ (i) $(x + \frac{1}{x})^7$, $[x \neq 0]$

7. The expression '$\sum_{p=0}^{19} \binom{19}{p} x^{n-p}(3y)^p$' is called <u>the binomial expansion of '$(x + 3y)^{19}$'</u>. The expression '$\binom{19}{11} x^{19-11}(3y)^{11}$' is called <u>the 12th term</u> of this expansion. For each of the following binomial exponentials, write, and simplify the indicated term.

<u>Sample</u>. 7<u>th</u> term of '$(2x - y)^{12}$'

<u>Solution</u>.

$$\binom{12}{6}(2x)^{12-6}(-y)^6 = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} 2^6x^6y^6$$
$$= 2^8\cdot 3\cdot 7\cdot 11x^6y^6$$

[$2^8\cdot 3\cdot 7\cdot 11 = 59136$. However, it is often simpler, particularly in the case of large numbers, to use a numeral which shows how the number is decomposed into its prime factors. (Recall that a prime integer is an integer, other than 1 or -1, which has no positive integer factors other than itself (or its opposite) and 1.)]

(a) 6<u>th</u> term of '$(1 - 2x)^{11}$' (b) 4<u>th</u> term of '$(3x + y)^{10}$'

(c) 8<u>th</u> term of '$(a^2 + bc)^9$' (d) 10<u>th</u> term of '$(3u - 6v)^{138}$'

(e) middle term of '$\left(\frac{x}{2} + \frac{y}{3}\right)^6$'

(continued on next page)

(f) second from last term of '$(p + 2q)^{97}$'

(g) constant term of '$(x + \frac{1}{x})^8$'

(h) 17th term of '$(6 + 10x)^{16}$'

* * *

We shall now use mathematical induction to prove the binomial theorem. In the proof we shall use methods like those used in the proof of the first boxed theorem in Exercise 7 of Part A. We shall also use the results of Exercises 5 and 7 of Part E. It will help if you review these things before reading further.

We wish to show that the property expressed by:

for every a and b,

$$(a + b)^{\cdots} = \sum_{p=0}^{\cdots} \binom{\cdots}{p} a^{\cdots - p} b^p$$

holds for every non-negative integer. To do this we shall show that both 0 and 1 have the property and that the property is hereditary over the set of positive integers.

Proof: (i) 0 has the property.

For, for every a and b, $(a + b)^0 = 1$ and

$$\sum_{p=0}^{0} \binom{0}{p} a^{0 - p} b^p = \binom{0}{0} a^0 b^0 = 1.$$

1 has the property.

For, for every a and b, $(a + b)^1 = a + b$ and

$$\sum_{p=0}^{1} \binom{1}{p} a^{1 - p} b^p = \binom{1}{0} a^{1 - 0} b^0 + \binom{1}{1} a^{1 - 1} b^1 = a + b$$

(ii) The property is hereditary over the set of positive integers.

For, suppose that, for a given integer $k \geq 1$, it is the case that for every a and b,

$$(a + b)^k = \sum_{p=0}^{k} \binom{k}{p} a^{k - p} b^p.$$

Then (for this k), for every a and b,

$$
\begin{aligned}
(a+b)^{k+1} &= (a+b)\cdot(a+b)^k \\
&= (a+b)\cdot\sum_{p=0}^{k}\binom{k}{p}a^{k-p}b^p \\
&= a\cdot\sum_{p=0}^{k}\binom{k}{p}a^{k-p}b^p + b\cdot\sum_{p=0}^{k}\binom{k}{p}a^{k-p}b^p \\
&= \sum_{p=0}^{k}\binom{k}{p}a^{(k-p)+1}b^p + \sum_{p=0}^{k}\binom{k}{p}a^{k-p}b^{p+1} \qquad \text{[See II, page 1-34.]} \\
&= \sum_{p=0}^{k}\binom{k}{p}a^{(k+1)-p}b^p + \sum_{q=1}^{k+1}\binom{k}{q-1}a^{k-(q-1)}b^q
\end{aligned}
$$

[In the second $\sum$-expression in the preceding step, replace 'p' by 'q - 1'.]

$$
\begin{aligned}
&= \left[\binom{k}{0}a^{k+1}b^0 + \sum_{p=1}^{k}\binom{k}{p}a^{(k+1)-p}b^p\right] \\
&\qquad + \left[\sum_{q=1}^{k}\binom{k}{q-1}a^{(k+1)-q}b^q + \binom{k}{k}a^0b^{k+1}\right]
\end{aligned}
$$

[It is in this step that we need the assumption that $k \geq 1$ [Why?].]

$$
\begin{aligned}
&= \binom{k}{0}a^{k+1}b^0 + \left[\sum_{p=1}^{k}\binom{k}{p}a^{(k+1)-p}b^p\right. \\
&\qquad \left. + \sum_{p=1}^{k}\binom{k}{p-1}a^{(k+1)-p}b^p\right] + \binom{k}{k}a^0b^{k+1} \\
&= \binom{k+1}{0}a^{k+1}b^0 + \sum_{p=1}^{k}\left[\binom{k}{p}+\binom{k}{p-1}\right]a^{(k+1)-p}b^p \\
&\qquad + \binom{k+1}{k+1}a^0b^{k+1}
\end{aligned}
$$

[By Exercise 5, Part E, $\binom{k}{0} = \binom{k+1}{0}$ and $\binom{k}{k} = \binom{k+1}{k+1}$; by I,

(continued on next page)

page 1-34, the bracketed expression in the preceding step can be condensed, as shown.]

$$(a + b)^{k+1} = \binom{k+1}{0} a^{k+1} b^0 + \sum_{p=1}^{k} \binom{k+1}{p} a^{(k+1)-1} b^p$$

$$+ \binom{k+1}{k+1} a^0 b^{k+1} \qquad \text{[By Exercise 7, Part E.]}$$

$$= \sum_{p=0}^{k+1} \binom{k+1}{p} a^{(k+1)-p} b^p .$$

That is, for every $k \geq 1$, if, for every a and b,

$$(a + b)^k = \sum_{p=0}^{k} \binom{k}{p} a^{k-p} b^p ,$$

then, for every a and b,

$$(a + b)^{k+1} = \sum_{p=0}^{k+1} \binom{k+1}{p} a^{(k+1)-p} b^p .$$

In other words, the property in question is hereditary over the set of positive integers. Since the property holds for 1, and is hereditary over the set of positive integers, it holds for every positive integer. Since it also holds for 0, it holds for every non-negative integer.

G. The Binomial Series.

You have seen that the binomial formula:

$$(a + b)^n = \sum_{p=0}^{n} \binom{n}{p} a^{n-p} b^p$$

holds for all real numbers a and b and for every integer $n \geq 0$. This suggests the question as to whether the formula (or some similar formula) holds for values of 'n' other than non-negative integers. For example, can we find a similar formula whose left side is '$(a + b)^{-1}$', or '$(a + b)^{\frac{1}{2}}$'?

The first difficulty to occur to one is that an expression which begins with '$\sum_{p=0}^{-1}$' or '$\sum_{p=0}^{\frac{1}{2}}$' is nonsense, and it is hard to see how the meaning of $\sum$-notation could be extended so that such expressions would make sense.

The boxed theorem on page 2-121 will suggest how this difficulty might be overcome. The theorem in question has the following consequence.

For every x such that $|x| < 1$,

$$(1 + x)^{-1} = \sum_{p=0}^{\infty} (-1)^p x^p .$$

[Hint: On page 2-121, replace 'a' by '1', 'x' by 'p + 1', and 'r' by '-x'.] For one thing, this suggests that we may be able to find "expansions" for those binomial exponentials in which the exponent symbol is a numeral for a number other than a non-negative integer, if we use infinite series rather than finite series. This suggestion will gain more force when we notice that the binomial formula itself can be replaced by an equation whose right side is an infinite series. To see this, recall that for every integer $n > 0$ and every integer p such that $0 < p \le n$,

$$(*) \qquad \binom{n}{p} = \frac{n(n-1)\ldots(n-p+1)}{1 \cdot 2 \cdot \ldots \cdot p} .$$

If we use (*) to define binomial coefficients corresponding to all positive integers n and p it is clear that, for $p > n$,

$$\binom{n}{p} = 0.$$ [One of the factors in the numerator has value 0.]

Hence, with this new definition of binomial coefficients (augmented by requiring, as before, that $\binom{n}{0} = 1$ for every integer $n \ge 0$), the binomial formula can be replaced by:

$$(a + b)^n = \sum_{p=0}^{\infty} \binom{n}{p} a^{n-p} b^p .$$ [Note '∞' in place of 'n'.]

This formula, as we know, holds for all values of 'a' and 'b' and all non-negative integer values of 'n'.

Referring now to the result that, for $|x| < 1$,

$$(1 + x)^{-1} = \sum_{p=0}^{\infty} (-1)^p x^p ,$$

we see that this can be incorporated into the Binomial Theorem if we define binomial coefficients in such a way that, for every integer $p \geq 0$,

$$\binom{-1}{p} = (-1)^p.$$

But this is precisely what formula (*) gives us if in it we replace 'n' by '-1'!

$$\binom{-1}{p} = \frac{(-1)(-2)\ \ldots\ (-p)}{1 \cdot 2 \cdot\ \ldots\ \cdot p}$$

$$= (-1)^p.$$

This now suggests that if we define binomial coefficients for every real number y and every integer $p \geq 0$ in such a way that

$$\binom{y}{0} = 1 \text{ and, for } p > 0, \ \binom{y}{p} = \frac{y(y-1)\ \ldots\ (y-p+1)}{1 \cdot 2 \cdot\ \ldots\ \cdot p} ,$$

then the formula

$$(a + b)^y = \sum_{p=0}^{\infty} \binom{y}{p} a^{y-p}\, b^p$$

may hold rather generally. [We know, at any rate, that it holds for all values of 'a', 'b' and 'y' such that the last is a non-negative integer, and for the value 1 of 'a' and -1 of 'y' if the value of 'b' is between -1 and 1.]

As a matter of fact it can be proved that for every a, b and y, such that $|b| < |a|$,

$$(a + b)^y = \sum_{p=0}^{\infty} \binom{y}{p} a^{y-p}\, b^p$$

[as well as for every a, b and y such that y is a non-negative integer].

The Binomial Theorem is, of course, a special case of this result. Aside from this, the most important consequence is the following.

> For every x and y such that $|x| < 1$,
>
> $$(1+x)^y = \sum_{p=0}^{\infty} \binom{y}{p} x^p .$$

For example, for every x such that $|x| < 1$,

$$\sqrt{1+x} = \sum_{p=0}^{\infty} \binom{\frac{1}{2}}{p} x^p .$$

That is, for every x such that $|x| < 1$, there is an N such that if $n > N$ then

$$\sum_{p=0}^{n} \binom{\frac{1}{2}}{p} x^p$$

is as close as you please to $\sqrt{1+x}$. Since $\binom{\frac{1}{2}}{0} = 1$ and, for every integer $p > 0$,

$$\binom{\frac{1}{2}}{p} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots(\frac{1}{2}-p+1)}{p!}$$

$$= \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\ldots(-\frac{2p-3}{2})}{p!} ,$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + ----.$$

Hence,

$$\sqrt{1.02} \stackrel{a}{=} 1 + 0.01 - 0.00005$$

$$= 1.00995 .$$

A better approximation is given by:

$$\begin{aligned}\sqrt{1.02} &\stackrel{a}{=} 1.00995 + 0.0000005 - 0.000000006 \\ &= 1.009950494\end{aligned}$$

[This approximation is, actually, correct to 9 decimal places. You can get as close an approximation as you wish to $\sqrt{1.02}$ by evaluating more terms of the binomial series.]

As another example let us find a decimal approximation to $1/\sqrt[3]{9}$.

$$\begin{aligned}1/\sqrt[3]{9} = 9^{-\frac{1}{3}} &= (8 + 1)^{-\frac{1}{3}} \\ &= 8^{-\frac{1}{3}}\left(1 + \frac{1}{8}\right)^{-\frac{1}{3}} \\ &= \frac{1}{2}\sum_{p=0}^{\infty}\binom{-\frac{1}{3}}{p}\left(\frac{1}{8}\right)^{p} \\ &\stackrel{a}{=} \frac{1}{2}\left(1 - \frac{1}{3}\cdot\frac{1}{8} + \frac{(-\frac{1}{3})(-\frac{1}{3} - 1)}{2}\left(\frac{1}{8}\right)^{2}\right. \\ &\qquad \left. + \frac{(-\frac{1}{3})(-\frac{1}{3} - 1)(-\frac{1}{3} - 2)}{2\cdot 3}\left(\frac{1}{8}\right)^{3}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{24} + \frac{1}{288} - \frac{1}{3328}\right) \\ &\stackrel{a}{=} 0.4808 .\end{aligned}$$

[This approximation is correct to 4 decimal places.]

1. Use the binomial series to find decimal approximations.

   (a) $\sqrt[4]{1.01}$ (b) $\sqrt{0.98}$ (c) $1/.996$

   (d) $\sqrt[3]{29}$ (e) $1/\sqrt[5]{29}$ (f) $(1.05)^{-6}$

2. Formulate a recursive definition of the binomial coefficients.